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H_∞ FIR Filter Gain Computation for Disturbed Systems using Linear Matrix Inequality

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***Abstract*—This paper presents the *a posteriori* H_∞ finite impulse response (H_∞ FIR) filter gain computation, the gain of the filter is obtained by using a linear matrix inequality (LMI) solution. The *a posteriori* H_∞ FIR filter is developed by minimizing the squared H_∞ norm of the weighted disturbance-to-error transfer function, where the weights are related to errors, the addition of those weights causes nonlinearities in the minimization problem, thus, this work presents an algorithm to avoid the nonlinearities and to solve the minimization problem by using only LMIs. The accuracy of the H_∞ FIR filter is shown against the unbiased FIR filter.**

***Index Terms*—H_∞ FIR filter, UFIR filter, Linear matrix inequalities.**

I. Introduction

State estimation is defined as all mathematical techniques that are required to estimate the state of some process, system, or object using its measurements [20]. Measurements are commonly conducted in the presence of noise, and also, the process could be involved into disturbances, so its desirable to have an accurate and precise estimator, preferably optimal, robust and unbiased. In the discrete-time state-space case, the state estimation can be conducted employing methods of optimal linear filtering based on the state-space equations of the system which states are desired to be estimated.

The finite impulse response (FIR) filters [20] are designed to provide state estimation at the discrete time index k over a finite horizon $[m, k]$ of N points where $m = k - N + 1$, with the following advantages: 1) bounded input bounded output (BIBO) stability [21], 2) numerical error reduction [10], and 3) higher robustness than in the Kalman filter (KF) [24]. For disturbed systems, the most effective robust FIR filters were obtained in the z-transform domain by minimizing estimation errors for maximized disturbances using the disturbance-to-error transfer function $\mathcal{T}$, therefore, such filters serve in linear time-invariant (LTI) systems.

The H_∞ filtering [5] [16] minimizes the H_∞ norm of $\mathcal{T}$ in the worst error case, which results in an energy-to-energy or $\mathcal{L}_2$-to-$\mathcal{L}_2$ robust structure. Different robust FIR filters were designed for disturbed systems in batch forms and using linear matrix inequalities (LMI) in [1], [2], [3], [11], and some early results were reported in [4], [9], [22], [23]. There were also developed robust filters and estimators for uncertain systems, mostly using the approach proposed in [6], [7], [12], but these solutions stay out the scope of this paper.

An important limitation of the above solutions is that measurement and initial errors are ignored, although, several other advanced robust FIR filters have recently developed in [13], [14], [15], [19], but H_∞ FIR filters for disturbed systems operating under measurement errors and initial errors have not been yet developed.

In this paper, we develop a robust *a posteriori* H_∞ FIR filter operating in disturbed systems with initial and measurement errors using LMI techiniques. Since the bounded real lemma (BRL) only applies if the LMI is linear, we modify the BRL for quadratic error covariance by introducing a new variable, and also, we propose a novel algorithm to compute the H_∞ FIR filter gain numerically using LMI and we present a numerical example for tuning the filter.

II. Extended LTI Discrete-Time State-Space Model

Consider a LTI system represented in discrete-time state-space with the following state and observation equations, respectively,

$$x_k = F x_{k-1} + E u_k + B w_k, \tag{1}$$
$$y_k = H x_k + v_k, \tag{2}$$

where $x_k \in \mathbb{R}^K$ is the state vector, $u_k \in \mathbb{R}^L$ is the input vector, $y_k \in \mathbb{R}^P$ is the observation vector, $w_k \in \mathbb{R}^M$ is the process noise and $v_k \in \mathbb{R}^P$ is the observation noise. The disturbance w_k and the measurement noise v_k are supposed to be zero mean, not obligatory Gaussian, mutually uncorrelated and norm-bounded.

Assume that $F \in \mathbb{R}^{K\times K}$, $E \in \mathbb{R}^{K\times L}$, $B \in \mathbb{R}^{K\times M}$ and $H \in \mathbb{R}^{P\times K}$ are known matrices.

The model in (1)-(2) cannot be used directly in FIR filtering and requires an extension on the horizon $[m, k]$ of N points, from $m = k - N + 1$ to k. This can be done if (1) is rewritten

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using the backward-in-time solutions as

$$x_k = Fx_{k-1} + Eu_k + Bw_k, \tag{3a}$$
$$x_{k-1} = Fx_{k-2} + Eu_{k-1} + Bw_{k-1}, \tag{3b}$$
$$\vdots$$
$$x_{m+2} = Fx_{m+1} + Eu_{m+2} + Bw_{m+2}, \tag{3c}$$
$$x_{m+1} = Fx_m + Eu_{m+1} + Bw_{m+1}, \tag{3d}$$
$$x_m = x_m + Eu_m + Bw_m, \tag{3e}$$

where the initial state x_m is supposed to be known and hence $u_m = 0$ and $w_m = 0$ in Eq. (3e). Then substituting (3d) into (3c) to modify (3c) for the initial state x_m and doing so until (3b) and (3a) are also modified for x_m allow extending (1) on $[m, k]$. By introducing the extended vectors

$$X_{m,k} = \begin{pmatrix} x_m^T & x_{m+1}^T & \cdots & x_k^T \end{pmatrix}^T \in \mathbb{R}^{NK}, \tag{4}$$
$$U_{m,k} = \begin{pmatrix} u_m^T & u_{m+1}^T & \cdots & u_k^T \end{pmatrix}^T \in \mathbb{R}^{NL}, \tag{5}$$
$$W_{m,k} = \begin{pmatrix} w_m^T & w_{m+1}^T & \cdots & w_k^T \end{pmatrix}^T \in \mathbb{R}^{NM}, \tag{6}$$

and referring to (3), the extended state equation can be written as

$$X_{m,k} = F_N x_m + S_N U_{m,k} + D_N W_{m,k}, \tag{7}$$

where the extended matrices are

$$F_N = \begin{pmatrix} I & F^T & \cdots & (F^{N-2})^T & (F^{N-1})^T \end{pmatrix}^T \in \mathbb{R}^{NK \times K}, \tag{8}$$

$$S_N = \begin{pmatrix} E & 0 & \cdots & 0 & 0 \\ FE & E & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F^{N-2}E & F^{N-3}E & \cdots & E & 0 \\ F^{N-1}E & F^{N-2}E & \cdots & FE & E \end{pmatrix} \in \mathbb{R}^{NK \times NL}, \tag{9}$$

$$D_N = \begin{pmatrix} B & 0 & \cdots & 0 & 0 \\ FB & B & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F^{N-2}B & F^{N-3}B & \cdots & B & 0 \\ F^{N-1}B & F^{N-2}B & \cdots & FB & B \end{pmatrix} \in \mathbb{R}^{NK \times NM}. \tag{10}$$

Similarly, the observation equation in (2) can be written as

$$y_k = Hx_k + v_k, \tag{11a}$$
$$y_{k-1} = Hx_{k-1} + v_{k-1}, \tag{11b}$$
$$\vdots$$
$$y_m = Hx_m + v_m. \tag{11c}$$

By substituting x_k, x_{k-1}, ..., x_m taken from (3) into (11) and assigning two vectors

$$Y_{m,k} = \begin{pmatrix} y_m^T & y_{m+1}^T & \cdots & y_k^T \end{pmatrix}^T \in \mathbb{R}^{NP}, \tag{12}$$
$$V_{m,k} = \begin{pmatrix} v_m^T & v_{m+1}^T & \cdots & v_k^T \end{pmatrix}^T \in \mathbb{R}^{NP}, \tag{13}$$

we obtain the extended observation equation

$$Y_{m,k} = H_N x_m + L_N U_{m,k} + G_N W_{m,k} + V_{m,k}, \tag{14}$$

in which the extended matrices are

$$H_N = \bar{H}_N F_N \in \mathbb{R}^{NP \times K}, \tag{15}$$
$$L_N = \bar{H}_N S_N \in \mathbb{R}^{NP \times NL}, \tag{16}$$
$$G_N = \bar{H}_N D_N \in \mathbb{R}^{NP \times NM}, \tag{17}$$

and matrix $\bar{H}_N$ is diagonal

$$\bar{H}_N = \begin{pmatrix} H & 0 & \cdots & 0 & 0 \\ 0 & H & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & H & 0 \\ 0 & 0 & \cdots & 0 & H \end{pmatrix} \in \mathbb{R}^{NP \times NK}. \tag{18}$$

The extended state-space equations described on (7) and (14) can be used to derive all kinds of linear convolution-based batch state estimators (filters, smoothers, and predictors) for given cost function, and the FIR filter will require a finite horizon $[m, k]$ of N points.

To design a FIR filter, the state x_k can be represented by the last row vector in (7) as

$$x_k = F^{N-1}x_m + \bar{S}_N U_{m,k} + \bar{D}_N W_{m,k}, \tag{19}$$

where the matrix $\bar{S}_N$ is the last row vector in S_N and so is $\bar{D}_N$ in D_N.

III. The H_∞ FIR Filtering

Before discussing H_∞ FIR filtering, recall that the H_2 FIR filter minimizes the squared Frobenius norm of the weighted error-to-error transfer function averaged over all frequencies [15]. Thereby, it provides optimal H_2 performance, but does not guarantee that possible peaks in the transfer function $\mathcal{T}$ will also be suppressed by averaging. Moreover, if the H_2 filter is not properly tuned, the peak errors in its output may grow due to bias errors, as in the Kalman Filter (KF) and Optimal-FIR (OFIR) filter.

The H_∞ filtering approach was developed to minimize the H_∞ norm of the disturbance-to-error (ς-to-ε) transfer function $\|\mathcal{T}\|_\infty = \sup \sigma_{\max}[\mathcal{T}(z)]$, where $\sigma_{\max}[\mathcal{T}(z)]$ is the maximum singular value of $\mathcal{T}(z)$. A feature of the H_∞ norm is that it minimizes the highest peak value of $\mathcal{T}(z)$ in the Bode plot. In H_∞ filtering, the induced H_∞ norm

$$\|\mathcal{T}\|_\infty = \sup_{\varsigma \neq 0} \frac{\|\mathcal{T}\varsigma\|_2}{\|\varsigma\|_2} = \sup_{\varsigma \neq 0} \frac{\|\varepsilon\|_2}{\|\varsigma\|_2} \tag{20}$$

of the ς-to-ε transfer function $\mathcal{T}$ [8] is commonly minimized, where the squared norms of the disturbance $\|\varsigma\|_2^2 = \sum_{i=m}^k \varsigma_i^* \varsigma$ and the estimation error $\|\varepsilon\|_2^2 = \sum_{i=m}^k \varepsilon_i^* \varepsilon$ are equal to their energies on $[m, k]$. Therefore, the H_∞ approach applies in both the time domain and the transform domain. Since $\|\mathcal{T}\|_\infty^2$ represents the maximum energy gain from ς to ε, then it follows that the H_∞ norm reflects the worst estimator case and its minimization results in a robust estimator. Moreover, for stable systems the H_∞ norm coincides with the $\mathcal{L}_2$ induced norm of the disturbance-to-error operator [18]. Therefore, it is also referred to as $\|\mathcal{T}\|_\infty = \|\mathcal{T}\|_{2,2}$.

In the standard formulation of H_∞ filtering [8], the robust H_∞ FIR filtering problem can be formulated as follows. Find the fundamental gain $\mathcal{H}_N$ for the H_∞ FIR filter to minimize $\|\mathcal{T}\|_\infty$, given by (20) on the horizon $[m, k]$, by solving the following optimization problem,

$$\mathcal{H}_N = \inf_{\mathcal{H}_N} \sup_{\varsigma \neq 0} \frac{\sum_{i=m}^k \varepsilon_i^T P_\varepsilon \varepsilon_i}{\sum_{i=m}^k \varsigma_i^T P_\varsigma \varsigma_i}, \tag{21}$$

where P_ε and P_ς are some proper weights. Since closed-form solutions for (21) can be found only in some special cases, consider the following problem

$$\mathcal{H}_N \Leftarrow \sup_{\varsigma\neq 0} \frac{\sum_{i=m}^{k} \varepsilon_i^T P_\varepsilon \varepsilon_i}{\sum_{i=m}^{k} \varsigma_i^T P_\varsigma \varsigma_i} < \gamma^2, \tag{22}$$

which allows to define $\mathcal{H}_N$ numerically for a given small positive $\gamma > 0$ and develop suboptimal algorithms. Note that the factor γ^2, which indicates the fraction of the disturbance energy that goes into the estimator error, should preferably be small. But because γ^2 cannot be too small for stable estimators, its value should be constrained.

A. *The* a posteriori H_∞ *FIR Filter*

To derive the H_∞ FIR filter, the column matrix rule and the bounded real lemma (BRL) are needed.

Lemma 1: (Column matrix rule). Given a block column matrix $Z_{m,k} = \begin{pmatrix} z_m^T & z_{m+1}^T & \cdots & z_k^T \end{pmatrix}^T$ *specified on* $[m, k]$*. Its recursive form is [11]*

$$Z_{m,k} = A_w Z_{m-1,k-1} + B_w z_k, \tag{23}$$

using the following strictly sparse matrices,

$$A_w = \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad B_w = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I \end{pmatrix}. \tag{24}$$

Proof. The proof is self-obvious. □ □ □.

Lemma 2: (Bounded real lemma (filtering)). Given a state space model

$$x_k = F x_{k-1} + B w_k, \tag{25}$$

$$y_k = H x_k + D w_k, \tag{26}$$

Let $\gamma > 0$ *and* $S = HB + D$*. If there exists a matrix* $X > 0$ *such that the following LMI is soluble,*

$$\begin{pmatrix} -X^{-1} & F & B & 0 \\ F^T & -X & 0 & F^T H^T \\ B^T & 0 & -\gamma P_w & S^T \\ 0 & HF & S & -\gamma P_y^{-1} \end{pmatrix} < 0, \tag{27}$$

then the following inequality holds on $[m, k]$*,*

$$\frac{\sum_{i=m}^{k} y_i^T P_y y_i}{\sum_{i=m}^{k} w_i^T P_w w_i} < \gamma^2. \tag{28}$$

Proof. Consider the dissipativity inequality [17] on a Finite Horizon (FH) $[m, k]$,

$$V(x_k) - V(x_m) < \sum_{i=m}^{k} s(w_i, y_i), \tag{29}$$

where $V(x_k)$ is the Lyapunov (storage) function representing the energy stored in the system at k and $s(w_i, y_i)$ is a supply function representing the energy that is supplied to the system at i. Choose the storage function $V(x_k) = x_k^T K x_k$ and, referring to (28), assign $s(w_i, y_i) = \gamma^2 w_i^T P_w w_i - y_i^T P_y y_i > 0$ to be the supply function. Then rewrite (29) as

$$\sum_{i=m}^{k} (y_i^T P_y y_i - \gamma^2 w_i^T P_w w_i) + \sum_{i=m+1}^{k} (V(x_i) - V(x_{i-1})) < 0,$$

substitute y_i taken from (26) and x_i from (25), assign $S = HB + D$, go to

$$\sum_{i=m}^{k} [(HFx_{i-1} + Sw_i)^T P_y ((HFx_{i-1} + Sw_i)) - \gamma^2 w_i^T P_w w_i] + \sum_{i=m+1}^{k} (x_i^T K x_i - x_{i-1}^T K x_{i-1}) < 0,$$

note that values beyond $[m, k]$, namely at $m - 1$, are not available for FIR filtering, change the lower limit in the first sum to $m + 1$, unite all components in one sum, and come up with

$$\sum_{i=m+1}^{k} [(HFx_{i-1} + Sw_i)^T P_y ((HFx_{i-1} + Sw_i)) - \gamma^2 w_i^T P_w w_i + x_i^T K x_i - x_{i-1}^T K x_{i-1}] < 0. \tag{30}$$

To eliminate variables, rearrange the terms and rewrite (30) as

$$\sum_{m+1}^{k} \begin{pmatrix} x_{i-1} \\ w_i \end{pmatrix}^T \Theta \begin{pmatrix} x_{i-1} \\ w_i \end{pmatrix} < 0,$$

which is satisfied if the following LMI holds,

$$\Theta = \begin{pmatrix} F^T K F + F^T H^T P_y H F - K & F^T K B + F^T H^T P_y S \\ B^T K F + S^T P_y H F & B^T K B + S^T P_y S - \gamma^2 P_w \end{pmatrix} < 0. \tag{31}$$

Then decompose (31) as

$$\begin{pmatrix} -K & 0 \\ 0 & -\gamma^2 P_w \end{pmatrix} + \begin{pmatrix} F^T & F^T H^T \\ B^T & S^T \end{pmatrix} \begin{pmatrix} K & 0 \\ 0 & P_y \end{pmatrix} \begin{pmatrix} F & B \\ HF & S \end{pmatrix} < 0,$$

consider it as a Schur's complement, and represent with another inequality

$$\begin{pmatrix} -K & 0 & F^T & F^T H^T \\ 0 & -\gamma^2 P_w & B^T & S^T \\ F & B & -K^{-1} & 0 \\ HF & S & 0 & -P_y^{-1} \end{pmatrix} < 0. \tag{32}$$

Now multiply (32) from the left-hand and right-hand sides with the following matrices, respectively,

$$\begin{pmatrix} 0 & 0 & I & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}, \qquad \begin{pmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

and obtain

$$\begin{pmatrix} -K^{-1} & F & B & 0 \\ F^T & -K & 0 & F^T H^T \\ B^T & 0 & -\gamma^2 P_w & S^T \\ 0 & HF & S & -P_y^{-1} \end{pmatrix} < 0. \tag{33}$$

Finally multiply (33) from both sides with

$$\begin{pmatrix} \gamma^{0.5}I & 0 & 0 & 0 \\ 0 & \gamma^{-0.5}I & 0 & 0 \\ 0 & 0 & \gamma^{-0.5}I & 0 \\ 0 & 0 & 0 & \gamma^{0.5}I \end{pmatrix},$$

introduce a new variable $X = K/\gamma$, arrive at (27), and complete the proof.
□ □ □.

To obtain the *a posteriori* H_∞ FIR filter using lemma 2, consider the model in (19) and (14) with $U_{m,k} = 0$ and define the FIR estimate as

$$\begin{aligned} \hat{x}_k &= \mathcal{H}_N Y_{m,k} \\ &= \mathcal{H}_N H_N x_m + \mathcal{H}_N G_N W_{m,k} + \mathcal{H}_N V_{m,k}. \end{aligned} \tag{34}$$

Now, using (19) with $U_{m,k} = 0$ and (34), the estimation error ε_k can be transformed to

$$\varepsilon_k = \mathcal{B}_N x_m + \mathcal{W}_N W_{m,k} - \mathcal{V}_N V_{m,k}, \tag{35}$$

where the error residual matrices are defined by

$$\mathcal{B}_N = F^{N-1} - \mathcal{H}_N H_N, \tag{36}$$
$$\mathcal{W}_N = \bar{D}_N - \mathcal{H}_N G_N, \tag{37}$$
$$\mathcal{V}_N = \mathcal{H}_N. \tag{38}$$

Using the following matrix forms given by lemma 1,

$$W_{m,k} = A_w W_{m-1,k-1} + B_w w_k, \tag{39}$$
$$V_{m,k} = A_w V_{m-1,k-1} + B_w v_k, \tag{40}$$

where matrices A_w and B_w are defined by (24).

Now introduce two augmented vectors $z_k = \begin{pmatrix} W_{m,k}^T & V_{m,k}^T & i_k^T \end{pmatrix}^T$, where $i_k = x_m$, and $\varsigma = \begin{pmatrix} w_k^T & v_k^T \end{pmatrix}^T$, and combine them in the following state-space model

$$z_k = \tilde{F}_\varsigma z_{k-1} + \tilde{B}_\varsigma \varsigma_k, \tag{41}$$
$$\varepsilon_k = \tilde{C}_\varsigma z_k, \tag{42}$$

in which the newly introduced block matrices have the form

$$\begin{aligned} \tilde{F}_\varsigma &= \begin{pmatrix} A_w & 0 & 0 \\ 0 & A_w & 0 \\ 0 & 0 & I \end{pmatrix}, \qquad \tilde{B}_\varsigma = \begin{pmatrix} B_w & 0 \\ 0 & B_w \\ 0 & 0 \end{pmatrix}, \\ \tilde{C}_\varsigma &= \begin{pmatrix} \mathcal{W}_N & -\mathcal{V}_N & \mathcal{B}_N \end{pmatrix}. \end{aligned} \tag{43}$$

Note that the strictly sparse matrices $\tilde{F}_\varsigma$ and $\tilde{B}_\varsigma$ can significantly reduce computational complexity.

An important property of the model in (41) and (42) follows immediately: all error residual matrices are combined into a new observation matrix $\tilde{C}_\varsigma$, which is thus completely responsible for the H_∞ filter performance.

B. LMI Based Algorithm for H_∞ FIR Filter Gain Computation

The state-space equations in (41) and (42) can't be used directly on the inequality given by lemma 2 due to weight P_y in (27) corresponds to P_ε in the H_∞ FIR Filtering problem, and $P_\varepsilon = P_k = \mathcal{E}\{\varepsilon_k \varepsilon_k^T\}$ is given by

$$P_\varepsilon = \mathcal{B}_N \chi_m \mathcal{B}_N^T + \mathcal{W}_N \mathcal{Q}_N \mathcal{W}_N^T + \mathcal{V}_N \mathcal{R}_N \mathcal{V}_N^T, \tag{44}$$

where $\chi_m = \mathcal{E}\left\{x_m x_m^T\right\} = \hat{x}_m \hat{x}_m^T + P_{\varepsilon_m}$, $\mathcal{Q}_N = \mathcal{E}\left\{W_{m,k} W_{m,k}^T\right\}$ and $\mathcal{R}_N = \mathcal{E}\left\{V_{m,k} V_{m,k}^T\right\}$, the residual matrices $\mathcal{B}_N$, $\mathcal{W}_N$, $\mathcal{V}_N$ are defined by (36)-(38). As can be seen in (44), P_ε is function of the gain $\mathcal{H}_N$, which is a desired variable for the minimization of γ in the inequality in (27), so the inequality is nonlinear with respect to the gain $\mathcal{H}_N$ due to the inversion of P_ε.

To solve this problem, first rewrite P_ε in (44) as

$$P_\varepsilon = \mathcal{A} - \mathcal{C}\mathcal{H}_N^T - \mathcal{H}_N \mathcal{C}^T + \mathcal{H}_N \mathcal{D} \mathcal{H}_N^T, \tag{45}$$

where the following matrices are introduced: $\mathcal{A} = F^{N-1}\chi_m (F^{N-1})^T + \bar{D}_N \mathcal{Q}_N \bar{D}_N^T$, $\mathcal{C} = F^{N-1}\chi_m H_N^T + \bar{D}_N \mathcal{Q}_N G_N^T$, $\mathcal{D} = H_N \chi_m H_N^T + \Omega_N$, and $\Omega_N = G_N \mathcal{Q}_N G_N^T + \mathcal{R}_N$. Then decompose (45) as:

$$P_\varepsilon = \begin{pmatrix} I & \mathcal{H}_N \end{pmatrix} \begin{pmatrix} \mathcal{A} & -\mathcal{C} \\ -\mathcal{C}^T & \mathcal{D} \end{pmatrix} \begin{pmatrix} I \\ \mathcal{H}_N^T \end{pmatrix}, \tag{46}$$

introduce new auxiliary matrices

$$\tilde{H}_\varsigma = \begin{pmatrix} I & \mathcal{H}_N \end{pmatrix}, \tag{47}$$
$$P_\mathcal{J} = \begin{pmatrix} \mathcal{A} & -\mathcal{C} \\ -\mathcal{C}^T & \mathcal{D} \end{pmatrix}, \tag{48}$$

and rewrite P_ε in (46) as

$$P_\varepsilon = \tilde{H}_\varsigma P_\mathcal{J} \tilde{H}_\varsigma^T. \tag{49}$$

Now replace F, B, H, P_w and P_y in (31) with $\tilde{F}_\varsigma$, $\tilde{B}_\varsigma$, $\tilde{C}_\varsigma$, P_ς and P_ε respectively as

$$\begin{pmatrix} \tilde{F}_\varsigma^T K \tilde{F}_\varsigma + \tilde{F}_\varsigma^T \tilde{C}_\varsigma^T P_\varepsilon \tilde{C}_\varsigma \tilde{F}_\varsigma - K & \tilde{F}_\varsigma^T K \tilde{B}_\varsigma + \tilde{F}_\varsigma^T \tilde{C}_\varsigma^T P_\varepsilon \tilde{C}_\varsigma \tilde{B}_\varsigma \\ \tilde{B}_\varsigma^T K \tilde{F}_\varsigma + \tilde{B}_\varsigma^T \tilde{C}_\varsigma^T P_\varepsilon \tilde{C}_\varsigma \tilde{F}_\varsigma & \tilde{B}_\varsigma^T K \tilde{B}_\varsigma + \tilde{B}_\varsigma^T \tilde{C}_\varsigma^T P_\varepsilon \tilde{C}_\varsigma \tilde{B}_\varsigma - \gamma^2 P_\varsigma \end{pmatrix} < 0, \tag{50}$$

where P_ς is given by

$$P_\varsigma = \mathcal{E}\{\varsigma_k \varsigma_k^T\} = \begin{pmatrix} Q_k & 0 \\ 0 & R_k \end{pmatrix}. \tag{51}$$

Replace (49) into (50) as

$$\begin{pmatrix} \tilde{F}_\varsigma^T K \tilde{F}_\varsigma + \tilde{F}_\varsigma^T \tilde{C}_\varsigma^T \tilde{H}_\varsigma P_\mathcal{J} \tilde{H}_\varsigma^T \tilde{C}_\varsigma \tilde{F}_\varsigma - K & \tilde{F}_\varsigma^T K \tilde{B}_\varsigma + \tilde{F}_\varsigma^T \tilde{C}_\varsigma^T \tilde{H}_\varsigma P_\mathcal{J} \tilde{H}_\varsigma^T \tilde{C}_\varsigma \tilde{B}_\varsigma \\ \tilde{B}_\varsigma^T K \tilde{F}_\varsigma + \tilde{B}_\varsigma^T \tilde{C}_\varsigma^T \tilde{H}_\varsigma P_\mathcal{J} \tilde{H}_\varsigma^T \tilde{C}_\varsigma \tilde{F}_\varsigma & \tilde{B}_\varsigma^T K \tilde{B}_\varsigma + \tilde{B}_\varsigma^T \tilde{C}_\varsigma^T \tilde{H}_\varsigma P_\mathcal{J} \tilde{H}_\varsigma^T \tilde{C}_\varsigma \tilde{B}_\varsigma - \gamma^2 P_\varsigma \end{pmatrix} < 0. \tag{52}$$

Then decompose (52) as

$$\begin{pmatrix} -K & 0 \\ 0 & -\gamma^2 P_\varsigma \end{pmatrix} + \begin{pmatrix} \tilde{F}_\varsigma^T & \tilde{F}_\varsigma^T \tilde{C}_\varsigma^T \tilde{H}_\varsigma \\ \tilde{B}_\varsigma^T & \tilde{B}_\varsigma^T \tilde{C}_\varsigma^T \tilde{H}_\varsigma \end{pmatrix} \begin{pmatrix} K & 0 \\ 0 & P_\mathcal{J} \end{pmatrix} \begin{pmatrix} \tilde{F}_\varsigma & \tilde{B}_\varsigma \\ \tilde{H}_\varsigma^T \tilde{C}_\varsigma \tilde{F}_\varsigma & \tilde{H}_\varsigma^T \tilde{C}_\varsigma \tilde{B}_\varsigma \end{pmatrix} < 0, \tag{53}$$

introduce a new matrix

$$\begin{aligned} \tilde{J}_\varsigma = \tilde{H}_\varsigma^T \tilde{C}_\varsigma &= \begin{pmatrix} \mathcal{W}_N & -\mathcal{V}_N & \mathcal{B}_N \\ \mathcal{H}_N^T \mathcal{W}_N & -\mathcal{H}_N^T \mathcal{V}_N & \mathcal{H}_N^T \mathcal{B}_N \end{pmatrix} \\ &= \begin{pmatrix} \bar{D}_N - \mathcal{H}_N G_N & -\mathcal{H}_N & F^{N-1} - \mathcal{H}_N H_N \\ \mathcal{H}_N^T \bar{D}_N - \mathcal{H}_N^T \mathcal{H}_N G_N & -\mathcal{H}_N^T \mathcal{H}_N & \mathcal{H}_N^T F^{N-1} - \mathcal{H}_N^T \mathcal{H}_N H_N \end{pmatrix}, \end{aligned} \tag{54}$$

and replace (54) into (53) as

$$\begin{pmatrix} -K & 0 \\ 0 & -\gamma^2 P_\varsigma \end{pmatrix} + \begin{pmatrix} \tilde{F}_\varsigma^T & \tilde{F}_\varsigma^T \tilde{J}_\varsigma^T \\ \tilde{B}_\varsigma^T & \tilde{B}_\varsigma^T \tilde{J}_\varsigma^T \end{pmatrix} \begin{pmatrix} K & 0 \\ 0 & P_\mathcal{J} \end{pmatrix} \begin{pmatrix} \tilde{F}_\varsigma & \tilde{B}_\varsigma \\ \tilde{J}_\varsigma \tilde{F}_\varsigma & \tilde{J}_\varsigma \tilde{B}_\varsigma \end{pmatrix} < 0,$$

consider it as a Schur's complement, and represent with another inequality

$$\begin{pmatrix} -K & 0 & \tilde{F}_\varsigma^T & \tilde{F}_\varsigma^T \tilde{J}_\varsigma^T \\ 0 & -\gamma^2 P_\varsigma & \tilde{B}_\varsigma^T & \tilde{B}_\varsigma^T \tilde{J}_\varsigma^T \\ \tilde{F}_\varsigma & \tilde{B}_\varsigma & -K^{-1} & 0 \\ \tilde{J}_\varsigma \tilde{F}_\varsigma & \tilde{J}_\varsigma \tilde{B}_\varsigma & 0 & -P_{\mathcal{J}}^{-1} \end{pmatrix} < 0. \quad (55)$$

Now multiply (55) from the left-hand and right-hand sides with the following matrices, respectively,

$$\begin{pmatrix} 0 & 0 & I & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}, \qquad \begin{pmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

and obtain

$$\begin{pmatrix} -K^{-1} & \tilde{F}_\varsigma & \tilde{B}_\varsigma & 0 \\ \tilde{F}_\varsigma^T & -K & 0 & \tilde{F}_\varsigma^T \tilde{J}_\varsigma^T \\ \tilde{B}_\varsigma^T & 0 & -\gamma^2 P_\varsigma & \tilde{B}_\varsigma^T \tilde{J}_\varsigma^T \\ 0 & \tilde{J}_\varsigma \tilde{F}_\varsigma & \tilde{J}_\varsigma \tilde{B}_\varsigma & -P_{\mathcal{J}}^{-1} \end{pmatrix} < 0. \quad (56)$$

Inequality in (56) still nonlinear with respect to the gain $\mathcal{H}_N$ due to the quadratic terms in $\tilde{J}_\varsigma$ defined by (54). To solve this problem, introduce an auxiliary matrix $\mathcal{Z}$ such that

$$\mathcal{Z} > \mathcal{H}_N^T \mathcal{H}_N. \quad (57)$$

Now rewrite (57) as

$$\mathcal{Z} - \mathcal{H}_N^T \mathcal{H}_N > 0. \quad (58)$$

If the Schur complement is used, (58) can be equivalently replaced with the LMI as

$$\begin{pmatrix} \mathcal{Z} & \mathcal{H}_N^T \\ \mathcal{H}_N & I \end{pmatrix} > 0. \quad (59)$$

Now. let's redefine $\tilde{J}_\varsigma$ in (54) using the new variable $\mathcal{Z}$ to replace the quadratic terms $\mathcal{H}_N^T \mathcal{H}_N$,

$$\tilde{J}_\varsigma = \begin{pmatrix} \bar{D}_N - \mathcal{H}_N G_N & -\mathcal{H}_N & F^{N-1} - \mathcal{H}_N H_N \\ \mathcal{H}_N^T \bar{D}_N - \mathcal{Z} G_N & -\mathcal{Z} & \mathcal{H}_N^T F^{N-1} - \mathcal{Z} H_N \end{pmatrix}. \quad (60)$$

Also, inequality in (56) is nonlinear with respect to the symmetric positive-definite matrix K. To avoid the inversion of K, pre- and post-multiply the matrix (56) with

$$\begin{pmatrix} K & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

and obtain another form of the inequality in (56),

$$\begin{pmatrix} -K & K\tilde{F}_\varsigma & K\tilde{B}_\varsigma & 0 \\ \tilde{F}_\varsigma^T K & -K & 0 & \tilde{F}_\varsigma^T \tilde{J}_\varsigma^T \\ \tilde{B}_\varsigma^T K & 0 & -\gamma^2 P_\varsigma & \tilde{B}_\varsigma^T \tilde{J}_\varsigma^T \\ 0 & \tilde{J}_\varsigma \tilde{F}_\varsigma & \tilde{J}_\varsigma \tilde{B}_\varsigma & -P_{\mathcal{J}}^{-1} \end{pmatrix} < 0. \quad (61)$$

Inequality in (61) has the form of a LMI if $\tilde{J}_\varsigma$ defined by (60) is used.

The gain $\mathcal{H}_N$ of the *a posteriori* H_∞ FIR filter now can be determined by solving the following minimization problem,

$$\mathcal{H}_N = \min_{\mathcal{H}_N, \mathcal{Z}, K, \gamma^2} \gamma^2$$
$$\text{subject to (59), (61) and } \mathcal{Z} = \mathcal{H}_N^T \mathcal{H}_N. \quad (62)$$

The third constraint of the minimization problem in (62), can be obtained by using Algorithm 1 in which gamma is minimized in each iteration by increasing the trace of $\mathcal{Z}$ in each iteration, at the end of each iteration the trace of $\mathcal{Z}$ is compared with the trace of $\mathcal{H}_N^T \mathcal{H}_N$, if the difference between them is greater than a small threshold $\delta_0 > 0$, the routine is ended and the gain is obtained. The best candidate for initializing the minimization procedure is of course the UFIR filter gain $\hat{\mathcal{H}}_N = F^{N-1}(H_N^T H_N)^{-1} H_N^T$.

Algorithm 1: Algorithm for H_∞ FIR Filter Gain Computation

Data: δ_0, $\hat{\mathcal{H}}_N$, P_ς, $\mathcal{Q}_{m,k}$, $\mathcal{R}_{m,k}$
Result: $\mathcal{H}_N$
begin
  $\mathcal{H}_N = \hat{\mathcal{H}}_N$;
  $\mathcal{Z} = \hat{\mathcal{H}}_N^T \hat{\mathcal{H}}_N$;
  while $|\mathrm{tr}(\mathcal{Z}) - \mathrm{tr}(\mathcal{H}_N^T \mathcal{H}_N)| < \delta_0$ **do**
    $\mathcal{Z}_{\text{prev}} = \mathcal{Z}$;
    $\mathcal{H}_N = \min \gamma^2$ subject to (59), (61) and $\mathrm{tr}(\mathcal{Z}) > \mathrm{tr}(\mathcal{Z}_{\text{prev}})$;
  end
end

Using the gain $\mathcal{H}_N$, numerically determined by using Algorithm 1, the *a posteriori* H_∞ filtering estimate and error covariance for uncorrelated w_k, v_k, and x_m can be obtained as, respectively,

$$\hat{x}_k = \mathcal{H}_N Y_{m,k}, \quad (63)$$
$$P_k = \mathcal{B}_N \chi_m \mathcal{B}_N^T + \mathcal{W}_N \mathcal{Q}_N \mathcal{W}_N^T + \mathcal{V}_N \mathcal{R}_N \mathcal{V}_N^T, \quad (64)$$

where the error residual matrices are given by (36)-(38).

IV. Numerical Example

Consider a radar and it is desired to measure a distance d_k in meters to a car that moves in discrete time index k with constant velocity v_k in meters by seconds. The process equations can be written as

$$d_k = d_{k-1} + \tau \mathrm{v}_k + w_{1k},$$
$$\mathrm{v}_k = \mathrm{v}_{k-1} + w_{2k},$$

where w_{1k} is a random error in the distance, w_{2k} is the error in the velocity, and $\tau = t_k - t_{k-1}$. Now assign two states. The first state is the distance $x_{1k} = d_k$, the second state is the velocity $x_{2k} = \mathrm{v}_k$. This gives the state equations

$$x_{1k} = x_{1(k-1)} + \tau x_{2k} + w_{1k},$$
$$x_{2k} = x_{2(k-1)} + w_{2k}.$$

Next, assume that noise $w_k \sim N(0, \sigma_w^2)$ only affects the velocity and assign

$$x_k = \begin{pmatrix} x_{1k} \\ x_{2k} \end{pmatrix}, \quad F = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2\tau \\ 1 \end{pmatrix},$$
$$w_{2k} = w_k, \quad w_{1k} = \tau w_k. \quad (65)$$

Then write the state equation

$$x_k = F x_{k-1} + B w_k. \quad (66)$$

For measured distance (first state), write the observation equation

$$y_k = H x_k + v_k, \quad (67)$$

where $H = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and $v_k \sim N(0, \sigma_v^2)$ is the measurement noise. As can be seen, the disturbance w_k and the measurement noise are chosen to be white gaussian noises, this is because in this paper we only limit to test the computation of the H_∞ FIR filter gain, the robustness of the filter will be tested for future works.

Now extend the state space (66) and (67) on $[m, k]$ and using $\tau = 0.025\,s$, $\sigma_w^2 = 12\,m$, $\sigma_v^2 = 10\,m/s$ as parameters of the system, the *a posteriori* H_∞ FIR filter gain can now be determined numerically by using Algorithm 1, the estimate can be computed as $\hat{x}_k = \mathcal{H}_N Y_{m,k}$ and also a estimation with the UFIR filter with $N_{\text{opt}} = 20$ will be computed as reference.

The behavioral of Algorithm 1 could be seen graphically on Fig. 1. The minimization of γ as the trace of $\mathcal{Z}$ is increased is shown in Fig. 1(a), as can be seen, if the trace of $\mathcal{Z}$ is increased, γ takes lower values as it gets to a minimum, but Fig. 1(b) shows that minimum value of gamma is not necessary the minimum that we are searching for. Fig. 1(b) shows the comparison of the trace of $\mathcal{Z}$ and the trace of $\mathcal{H}_N^T \mathcal{H}_N$ as the trace of $\mathcal{Z}$ is increased, in this case both graphs are very similar with lower values of the trace of $\mathcal{Z}$ but there is a point where this trace is increased and the other one diverges, this is because at this point of the algorithm $\mathcal{Z} \neq \mathcal{H}_N^T \mathcal{H}_N$ and the third restriction in (62) isn't satisfied, so we have to stop the algorithm before both traces starts to diverge and obtain the H_∞ FIR filter gain.

Typical filtering errors are shown in Fig. 2, it can be inferred than UFIR filter is the one who gives the less accurate estimates, while the estimation using the gain computed with Algorithm 1 looks like giving better estimates. The RMSE for each filter is given in Table I, and, as can be graphically seen on Fig. 2, the UFIR filter is the less accurate. The H_∞ FIR filter using Algorithm 1 to compute the gain has lower errors than the UFIR, and because this filter operates under H_∞ performance, we expect that this filter will be almost as robust as the UFIR filter.

TABLE I
RMSEs PRODUCED BY THE FILTERS.

Filter	RMSE
UFIR	33.2906
H_∞ FIR	**31.6697**

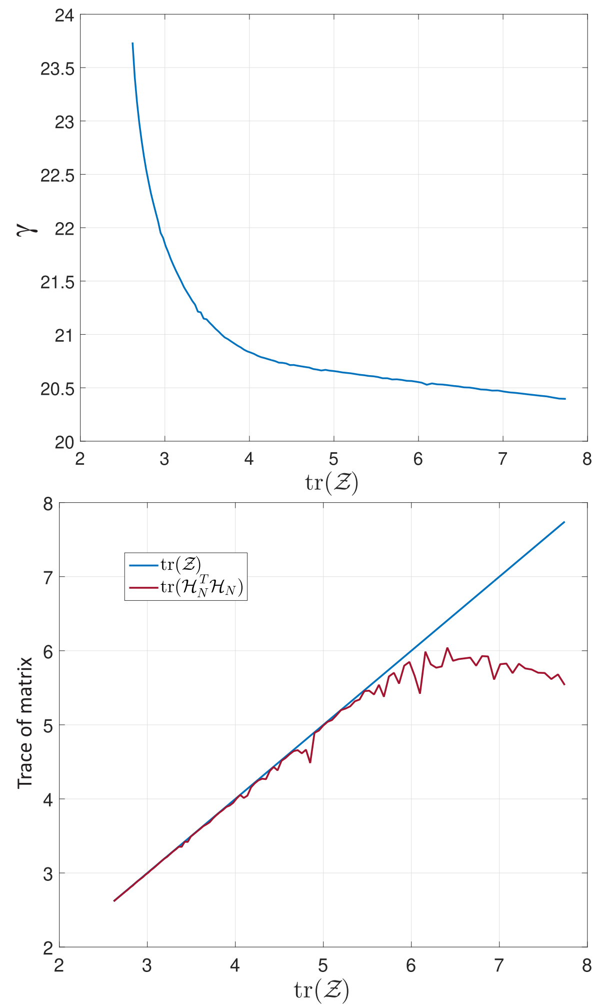

Fig. 1. Solving the minimization problem by Algorithm 1: (a) minimizing γ as the trace of $\mathcal{Z}$ is increased and (b) comparison between the trace of matrix $\mathcal{H}_N^T \mathcal{H}_N$ and the trace of $\mathcal{Z}$ as the trace of $\mathcal{Z}$ is increased.

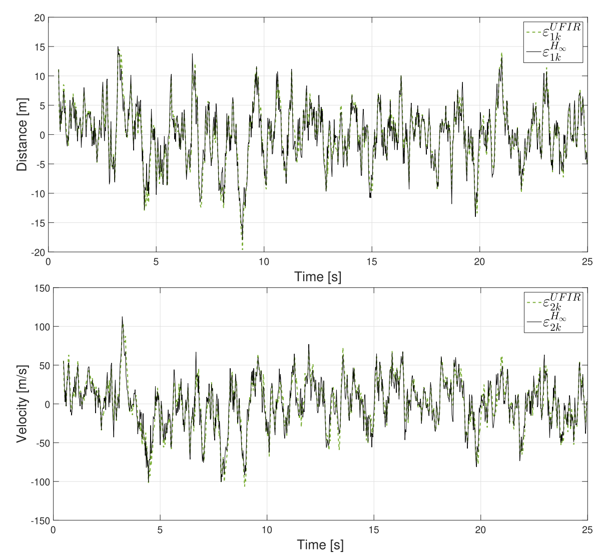

Fig. 2. Filtering errors produced by the filters in the example system for: (a) first state and (b) second state.

V. Conclusions

The cost function to minimize in the H_∞ FIR filter problem, cannot be minimized as a convex function so numerical approaches are needed. In this paper, the H_∞ FIR filter gain is obtained by using a LMI based algorithm, which must be done to avoid quadratic constraints and non linearities in the variables of the inequality. In this case, the H_∞ FIR gain is obtained one time without initial parameters working as an unbiased FIR filter.

The numerical example seen in this paper was only for filter tuning and test the new algorithm, due to this reason, the numerical example was taken with White Gaussian Noise (WGN) as process and measurement noise, for future work, the H_∞ performance will be tested under Colored Process Noise (CPN) and Colored Measurement Noise (CMN), to test the robustness of the filter, comparing the robustness with the UFIR Filter, OUFIR Filter and also the H_2-OUFIR Filer.

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