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# BLF-based Robust Navigation of Nonholonomic Mobile Robots Subject to Unmatched Disturbances

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***Abstract*—Typically, the deformation of the tire in contact with the road has been neglected, thus a constant contact area $A_c$ with a fixed infinitesimal pressure contact point $P_c$ is assumed. However, wheel shows lateral and radial deformation, resulting in a non-constant contact area $A_{vc}$ with a $P_{vc}(t) \neq P_c$. Consequently, there arises endogenous forces when pivoting w.r.t. $P_{vc}$ that may provoke skidding and/or slipping phenomena for nonholonomic (NH) vehicles, such as a car or a wheeled mobile robot (WMR). This phenomena may compromise tracking tasks of NH-WMR. Given the difficulty of measuring $P_{vc}$, these endogenous unmodelled forces have been cast wrongly as exogenous forces that appears as disturbances at a dynamic level, hence neglecting them as kinematic disturbances, which may lead to further contradiction in the models. In this paper, a Barrier Lyapunov Function (BLF) approach is addressed to handle robust velocity tracking for an *affine* differential kinematic NH-WMR model subject to unmatched disturbances, which arises when considering $P_{vc}(t)$ at dynamic level. The closed-loop system yields convergence to an invariant set within the imposed barriers around a nominal desired reference provided by a smooth fuzzy velocity field. Representative simulations show that the mobile robot remains within the barriers width wich stands for the tolerance region of lateral disturbance velocity.**

***Index Terms*—Wheeled mobile robot, Barrier Lyapunov function, Velocity field, Backstepping.**

## I. Introduction

The WMR is modelled as an underactuated system subject to a nonholonomic constraints that enforces only forward and steering motions, that is, lateral velocities are annihilated because their distribution is non involutive, while underactuation arises from accounting for less actuators than the degrees of freedom (DoF).These apparently uncommon phenomena are in fact present in our daily lives since a car or a bicycle are examples of such involved motion. But there is more, there exists an additional effect largely neglected in the literature: an ideal rotation is assumed at the constant contact point $P_c$ between the wheel and the road, which hardly applies in practice due to wheel deformation. Consequently, in the realm of a physical setting, the NH-WMR dynamics is subject to an additional endogenous force $\tau_e$ stemmed from the fact that the contact point $P_c$ moves, thus there arises $P_{vc}(t)$ due to wheel deformation, which gives rise to $\tau_e$ from the cross product with respect to the wheel friction forces. This phenomena is exacerbated at high speeds, sharp turns or in low friction surfaces from road conditions, where larger deformation wheel appears, thus skidding and slipping may occur. Consequently, the velocity of the contact point $P_{vc}(t)$ is no longer null, hence, not only the nonholonomic assumption is *compromised*, but stability is at risk.

From the perspective of singular perturbation formulation, it can be seen a two time scales dynamics: a fast one at position level and a slower one from wheel interaction forces. In [1], it is presented a kinematic NH-WMR subject to skidding and slipping where slipping is cast as a matched (additive) disturbance while skidding turns out as an unmatched disturbance. However, dealing with sliding is a rather complex task since it arise from highly nonlinear traction forces, demanding a robust model-free controller. In [2], a novel adaptive trajectory tracking is introduced for the NH-WMR subject to longitudinal slipping and state constraints. Based on a linearized dynamic model, a model-free controller is proposed using the approximation capabilities of fuzzy logic, providing safety of operations in presence of slipping. The experimental validation is ingenious and illustrative: whereas longitudinal slipping arise as a delay, lateral skidding compromises the NH-WMR orientation, thus it remains unclear how to address these effects at kinematic level.

The kinematic BLF-based design imposes a challenge because underactuation is not in feedback form. In [3], a BLF neural network kinematic controller was presented using a chained form where conventional disturbances were included in the dynamic model; however, skidding and slipping where not considered since a disturbance-free kinematic model is used. In [4], a novel but discontinuous relay-based finite-time controller was presented for the kinematic NH-WMR in chained form, subject to matched disturbances. Although experimental validation was presented, the disturbance arise from numerical evaluation, rather than an experimental setting. Furthermore, unmatched disturbances were not considered.

In this paper, a robust BLF-based control scheme is proposed for kinematic chained form of a NH-WMR subject to disturbances. State constraints are imposed to guarantee practical tracking within the barrier regions. In addition, a fuzzy velocity field is proposed as the smooth desired reference such that the robot converges smoothly despite unmatched disturbances, and UGES is obtained when disturbances are

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non persistent.

The rest of this paper is organized as follows. Section II gives the preliminaries and the problem design statement. Design of control, and velocity field are given in Sections III and IV. Section V evaluates the numerical performance, and finally, some concluding remarks are presented in Section VI.

## II. Preliminaries and problem statement

The following technical results are needed to facilitate the derivation of the main result.

### A. Preliminaries

**Definition 1.** *(Barrier Lyapunov Function-BLF [5]). A BLF is a scalar function $V(x)$, defined with respect to system $\dot{x} = f(x)$ in an open region $D \subset \mathbb{R}^n$ containing the origin, that has the property $V(x) \to \infty$ as $x$ approaches the boundary of the open region $\partial D$ and $V(x) \leq b$, $\forall t \geq 0$.*

**Lemma 1.** *[5] For any positive constant $k_{b1}$, the following inequality holds for $x$ in the interval $|x| < k_{b1}$:*

$$\ln\left(\frac{k_{b1}^2}{k_{b1}^2 - x_1^2}\right) < \frac{x_1^2}{k_{b1}^2 - x_1^2}. \tag{1}$$

### B. The (2,0) WMR Differential Kinematic Model

Consider the type (2,0) WMR governed by the unicycle kinematics subject to skidding and slipping [1], for $x, y$ the Cartesian (operational) coordinates, $\theta$ the orientation, $v, \omega$ the exogenous control inputs representing translational and wheel angular velocities, respectively, then

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}\cos\theta & 0\\ \sin\theta & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}v\\ \omega\end{bmatrix} + \begin{bmatrix}\cos\theta & 0 & \sin\theta\\ \sin\theta & 0 & -\cos\theta\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\Delta_v\\ \Delta_\omega\\ d\end{bmatrix} \tag{2}$$

where the terms $\Delta_v, \Delta_\omega$ stand for additive disturbance representing slippage velocity, and $d = d(t)$ models a time varying unmatched disturbance representing skidding velocity; and $q = [\xi, \theta]^T \in SE(2)$ is the pose vector, where $\xi = [x, y]^T \in \mathbb{R}^2$. System (2) can be written as follows

$$\dot{q} = B(q)u + D(q)\Delta \tag{3}$$

which represents a forced flowless ordinary differential equation subject to a disturbance $\Delta = [\Delta_\omega, \Delta_v, d(t)]^T$, for control $u = [v, \omega]^T \in \mathbb{R}^2$, and $q \in \mathbb{R}^3$.

Notice that (3) stands for a disturbed underactuated flowless differential kinematics since $\text{rank}(B) = 2$ but $\text{rank}(D) = 3$. Thus, this system has a non-involutive distribution, meaning that it is not integrable in $\mathbb{R}^3$ [6], due to the fact that it is subject to a nonintegrable equality (nonholonomic constraint) involving the velocity vector $[v, \omega]$, that is, for $\Delta = 0$ and at rest $\dot{q} = 0$, (3) yields $B(q)u = B(q)[v, \omega]^T = 0$. In other words, the span of $B(q)$ annihilates a non-zero vector $[v, \omega]$.

**Assumption 1.** *Kinematic disturbances $\Delta = (\Delta_\omega, \Delta_v, d(t))$ are assumed piecewise continuous in $t$ and locally Lipchitz in $q$, i.e. $\|\Delta_\omega\| \leq \delta_\omega,\ \|\Delta_v\| \leq \delta_v,\ \|d(t)\| \leq \delta_d$, for some positive bounded constant $\delta_v, \delta_\omega, \delta_d$.*

### C. Problem Statement

The canonical NH-WMR does not satisfy linear controllability, but [6] shows that it satisfies small-time local accessibility for the system (3) transformed into the disturbed chained-form model. To see this, notice that system (3) can be written into this form:

$$\begin{aligned}\dot{x}_1 &= \nu_1 + \Delta_\omega,\\ \dot{x}_2 &= \nu_2 + \Delta_v,\\ \dot{x}_3 &= x_2\nu_1 + d(t).\end{aligned} \tag{4}$$

for the chained state $x = [x_1, x_2, x_3]^T$ with the following transformation of $[x, y, \theta]$

$$\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\ \cos\theta & \sin\theta & 0\\ \sin\theta & -\cos\theta & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ \theta\end{bmatrix}, \tag{5}$$

and using the following forward velocities $[\nu_1, \nu_2]^T$

$$\begin{bmatrix}v\\ \omega\end{bmatrix} = \begin{bmatrix}x_3 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\nu_1\\ \nu_2\end{bmatrix} \tag{6}$$

In this paper, we adopt this disturbed chained form to formally tackle a control design. To this end, system (4) suggests to design nominal reference trajectories as follows

$$\begin{aligned}\dot{x}_{1_d} &= \nu_{1_d}\\ \dot{x}_{2_d} &= \nu_{2_d}\\ \dot{x}_{3_d} &= x_{2d}\nu_{1d}\end{aligned} \tag{7}$$

Using (4) and (7), there arise the following error dynamics

$$\dot{e}_1 = \nu_{1_d} - \nu_1 \tag{8}$$
$$\dot{e}_2 = \nu_{2_d} - \nu_2 \tag{9}$$
$$\dot{e}_3 = x_{2_e}\nu_{1_d} - x_2\dot{x}_{1_e} - d(t) \tag{10}$$

where $e_i := x_{i_d} - x_i$ is the tracking error. Finally, we need the following assumption, which we show to be consistent with the design of the desired reference using a smooth velocity field $\mathcal{V}$.

**Assumption 2.** *The signals* (7) *and its derivative $\dot{\nu}_{1_d}$ are uniformly continuous. Additionally, the forward reference velocity $\nu_{1_d}$ is persistent, i.e. there exists $\underline{\varepsilon}_{\nu_{1,d}},\ \overline{\varepsilon}_{\nu_{1,d}} \in \mathbb{R}^+$ s.t. $\underline{\varepsilon}_{\nu_{1,d}} \leq \nu_{1_d} \leq \overline{\varepsilon}_{\nu_{1,d}}$.*

At this point, the control problem can be stated as follows. *Given a NH-WMR subject to bounded external disturbances* (3)*, design the control input $u = [v, \omega]^T$ such that $(q, \dot{q}) \to (q^d, \mathcal{V})$ locally uniformly asymptotically, ensuring the boundedness of all closed-loop signals, and that state $x_2, x_3$ constrained within the barriers*

$$|x_2| \leq k_{s_2}, \quad |x_3| \leq k_{s_3} \tag{11}$$

*where $k_{s_2}, k_{s_3}$ are positive constants.*

### D. Further Analysis

Aiming at providing a kinematic controller for a realistic setting, that is, considering the consequences of unmatched kinematic disturbances due to skidding and slipping, a further examination of (3) is needed to elucidate some additional properties. Let us analyze the kinematic model subject to disturbances (3) under the nonholonomic-affine constraint framework [7]: we shall show that regardless of the control design approach, matched disturbances can be disregarded into the differential kinematic model, as their effect can be compensated by a robust dynamic controller, but it is mandatory to consider unmatched disturbances. This observation becomes instrumental, so let us present this result and its proof, [7].

**Proposition 1** ( [7])**.** *The disturbed NH-WMR* (3) *is subject to constraints of the form* $A(q)\dot{q} - s(q) = 0$*, where* $A(q) = \begin{pmatrix} -\sin\theta & \cos\theta & 0 \end{pmatrix}$*. Thus,* $d(t)$ *arise as an affine constraint in velocity.*

*Proof.* Let the disturbance-free kinematic model (3) be $\dot{q} = B(q)u$, where input matrix $B(q) = \begin{bmatrix} g_1 & g_2 \end{bmatrix}$ for input vector fields $g_i(q)$. Then the disturbance projection in (2) can be rewritten as $D(q) = \begin{pmatrix} B(q) & -g_3 \end{pmatrix}$ , where $g_3 = [g_1, g_2]$ is the Lie bracket of the input vector fields. Thus, $D(q)$ spans the accessibility distribution of $\dot{q} = B(q)u$, [6]. Consider the annihilator of the non integrable distribution that arise from the pure rolling constraint $A(q) = \begin{pmatrix} -\sin\theta & \cos\theta & 0 \end{pmatrix}$, that remains orthogonal to the input (control) vector fields $g_i(q)$ i.e. $A(q)\{g_i\} = 0$. Now, applying $A(q)$ to both sides of (2), we get $A(q)\dot{q} = s(q)\Delta$. Finally, as $g_3 = -A^\intercal(q)$ we get

$$A(q)\dot{q} = A(q)A^\intercal(q)d(t) = \left(\sin^2\theta + \cos^2\theta\right)d(t) = d(t),$$

showing that the skidding velocity is in the span of the annihilator of $\dot{q}$. Therefore, it can be concluded that the kinematic model (3) defines a nonholonomic affine system, [7], since its reaction-annihilator distribution [8] allows the lateral skidding velocity to arise as an velocity term removed from the kernel of the non-integrable distribution $A(q)$. □

**Remark 1.** *Notice that* $d(t)$ *indeed appears as motions due to endogenous reaction forces arising from the interaction of the wheel deformation in contact with the rigid road; therefore, it is reasonable to argue that, different from the slipping velocities* $\Delta_{v,\omega}$*, the skidding velocity* $d(t)$ *acquires now a clear geometrical interpretation. For instance, Proposition 1 suggests that regardless of linearization (such as the look-ahead-like transformation [2], [9]), dealing with unmatched disturbances at a kinematic level is a problem that must be addressed separately.*

## III. Control Design

### A. Rationale to Design a Backstepping Control

Although differential kinematic model (3) leads apparently to a simplified representation of the dynamical (2,0) WMR, it preserves the properties of the disturbed original system; in addition, note that the chained form is in a suitable representation for the backstepping algorithm to address a robust navigation scheme. In virtue of Proposition 1, the control design problem can be simplified to the design the velocities $u = (v, \omega)$ considering only unmatched disturbances. Firstly, notice from (5) that orientation $x_1 = \theta$ remains unchanged under the chained form transformation. Hence, from (4), orientation error subsystem $(e_1, \dot{e}_1)$ can be uniquely stabilized by means of the steering velocity control input $\nu_1 = \omega$. Henceforth, [4], we aim at designing a faster convergence of orientation coordinate in comparison to the position coordinates convergence to address accessibility to the desired path. Inspired by [3], the strict feedback form of the position subsystem (4) is considered to the backstepping control design [5] to enforce state constraints as barriers (11) whereas underactuation is solved. However, our approach differs from [5] in a subtle way: we aim at exploiting the geometric interpretation of the strict state constraints (see Fig. 1) to bound tracking errors due to the unmatched lateral skidding disturbance within an *arbitrary fixed safety region around the desired trajectory*.

### B. Barriers as State Constraints

Let $y = e_3$ be the output of the underactuated system (8)-(10), and consider the *backstepping-like* error transformation

$$\begin{aligned} z_1 &= x_{3_d} - x_3 = e_3, \\ z_2 &= \alpha_1 - x_2, \end{aligned} \tag{12}$$

derivating (12), we obtain its error dynamics, using (5),

$$\begin{aligned} \dot{z}_1 &= (e_2 + z_2 - \alpha_1)\,\nu_{1_d} + x_2\dot{e}_1 - d(t), \\ \dot{z}_2 &= \dot{\alpha}_1 - \nu_2. \end{aligned} \tag{13}$$

Now, let the strict state constraints be reformulated as follows

$$|z_1| < k_{1e}, \quad |z_2| < k_{2e}, \tag{14}$$

where $k_{1e} = X_3 + k_{s3},\ k_{2e} = \overline{\alpha} + k_{s2}$, and $|x_{3_d}| < X_3, |\alpha| < \overline{\alpha}$, for some $X_3, \overline{\alpha} \in \mathbb{R}^+$ that are the upper bounds of $x_{3_d},\ \alpha$, respectively. Thereby, barriers width $X_3, \overline{\alpha}$ are the *vicinity* around the transformed trajectory of all admissible errors, as shown in Fig. 1.

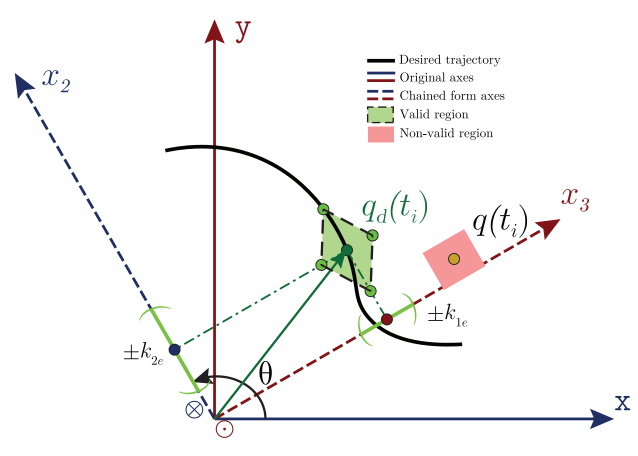



Fig. 1. Geometrical interpretation of the strict bound $k_{1e}, k_{2e}$.

### C. Barrier Lyapunov Function-based Backstepping Design

The BLF backstepping control design is presented into two steps: firstly, the steering velocity control $\nu_1$ is designed to guarantee accessibility of the reference trajectory, followed by the design of the auxiliary stabilizing function $\alpha$, and then finally the forward velocity input $\nu_2$.

*Step 1:* Consider the Lyapunov candidate function $V_1(t, e_1) = e_1^2/2$ whose time derivative along the solutions of (8) yields $\dot{V}_1 = e_1(\nu_{1_d} - \nu_1)$. Let the steering velocity control $\nu_1$ be

$$\nu_1 := \nu_{1_d} + k_0 e_1, \tag{15}$$

hence,

$$\dot{V}_1 = -k_0 e_1^2, \tag{16}$$

yielding exponential convergence of $e_1$.

*Step 2:* Consider the symmetric candidate log-type BLF, for $V_2 = V_2(z_1, z_2)$,

$$V_2 = \frac{1}{2}\ln\left(\frac{k_{1e}^2}{k_{1e}^2 - z_1^2}\right) + \frac{1}{2}\ln\left(\frac{k_{2e}^2}{k_{2e}^2 - z_2^2}\right) \tag{17}$$

whose time derivative along the solutions of (13) is given by

$$\begin{aligned}\dot{V}_2 \quad = \quad & \frac{z_1(x_{2_d} - \alpha_1)\nu_{1_d}}{k_{1e}^2 - z_1^2} + \frac{z_1(x_2\dot{e}_1 - d(t))}{k_{1e}^2 - z_1^2} + \frac{z_1 z_2}{k_{1e}^2 - z_1^2}\nu_{1_d} \\ & + \frac{z_2(\dot{\alpha}_1 - \nu_2)}{k_{2e}^2 - z_2^2}.\end{aligned} \tag{18}$$

Now, let the *auxiliary stabilizing function* $\alpha_1$ and the *forward velocity control law* $\nu_2$ be designed as follows

$$\alpha_1 := x_{2,d} + k_1 z_1 \nu_{1_d} \tag{19}$$

$$\nu_2 := \dot{\alpha}_1 + z_1\nu_{1_d}\frac{k_{2e}^2 - z_2^2}{k_{1e}^2 - z_1^2} + k_2 z_2 \nu_{1_d}^2. \tag{20}$$

Using (19)-(20), (18) becomes

$$\begin{aligned}\dot{V}_2 \quad = \quad & -k_2\nu_{1_d}^2\frac{z_2^2}{k_{2e}^2 - z_2^2} - k_1\nu_{1_d}^2\frac{z_1^2}{k_{1e}^2 - z_1^2} - \frac{z_1 d(t)}{k_{1e}^2 - z_1^2} \\ & + x_2\dot{e}_1\frac{z_1}{k_{1e}^2 - z_1^2}.\end{aligned} \tag{21}$$

### D. Main Result 1

We state the main result for the BLF-based kinematic controller in the following Proposition.

**Proposition 2.** *Consider the Assumptions 1 and 2, and let* $\tilde{\boldsymbol{e}} := (e_1, z_1, z_2)$ *be the extended error, and* $\mathcal{Z} := \left\{\tilde{\boldsymbol{e}} \in \mathbb{R}^3 : |z_1| < k_{e1}, |z_2| < k_{e2}\right\}$ *be an open set. For any* $\tilde{\boldsymbol{e}}(0) \in \mathcal{Z}$*, the equilibrium point* $\tilde{\boldsymbol{e}}_i = 0$ *of the closed-loop system* (4) *and* (13) *under control laws* (15) *and* (20) *is uniformly locally asymptotic stable.*

*Proof.* We first need to prove that there exists a nonempty subset $\Omega_{\tilde{e}} \subset \mathcal{Z}$ that contains the origin, such that $\forall(t_0, \tilde{\boldsymbol{e}}_0) \in \mathbb{R}^+ \times \mathcal{Z}$ the auxiliary stabilizing function $\alpha(t)$ is well defined on the maximal time interval $(t_0, T]$ of the solution of $\dot{\tilde{\boldsymbol{e}}}$. First, let the Lyapunov function candidate be

$$V(t, \tilde{\boldsymbol{e}}) = V_1 + V_2 \tag{22}$$

whose time derivative becomes

$$\begin{aligned}\dot{V} = & -k_0 e_1^2 - k_2\nu_{1_d}^2\frac{z_2^2}{k_{2e}^2 - z_2^2} - k_1\nu_{1_d}^2\frac{z_1^2}{k_{1e}^2 - z_1^2} \\ & - \frac{z_1 d(t)}{k_{1e}^2 - z_1^2} + \frac{z_1 x_2 \dot{e}_1}{k_{1e}^2 - z_1^2}.\end{aligned}$$

From (16), and by virtue of *Barbalat's Lemma*, $\ddot{V}_1 \le -2k_0^2 e_1^2$, we can conclude that $e_1$ is uniformly ultimately bounded with exponential convergence, thus $\dot{e}_1 \to 0$, which leads to (23) being written as

$$\dot{V} = -k_0 e_1^2 - k_2\nu_{1_d}^2\frac{z_2^2}{k_{2e}^2 - z_2^2} - k_1\nu_{1_d}^2\frac{z_1^2}{k_{1e}^2 - z_1^2} - \frac{z_1 d(t)}{k_{1e}^2 - z_1^2}.$$

Notice that $\dot{V}$ is already negative semi-definite *iff* $z_1 \in \mathcal{Z}$ and $\|z_1\| > \frac{\delta_d}{k_1 \varepsilon^*_{\nu_{1,d}}} =: \epsilon_{z_1}$, where $\varepsilon^*_{\nu_{1,d}} \ge \nu_{1,d}^2$, implying that any asymptotic solution of the extended error system $\dot{\tilde{\boldsymbol{e}}}$ remains in the invariant set

$$\Omega_{\tilde{e}} := \left\{\tilde{\boldsymbol{e}} \in \mathbb{R}^3 : \|z_1\| \le \epsilon_{z_1}, z_2 = x_{1,e} = 0\right\} \subset \mathcal{Z}, \tag{23}$$

Finally, to prove uniform stability from Lemma 1 we can compute the upper bound of (22) as follows

$$2V \le \nu_{1d}^2\left(\frac{k_1 z_1^2}{k_{1e}^2 - z_1^2} + \frac{k_2 z_2^2}{k_{2e}^2 - z_2^2}\right) + k_0 e_1^2, \tag{24}$$

$$\le -\dot{V} + \frac{z_1 d(t)}{k_{1e}^2 - z_1^2}. \tag{25}$$

By virtue of Assumptions 1 and 2 we can drop time dependency, therefore

$$\dot{V}(\tilde{\boldsymbol{e}}) \le -2V(\tilde{\boldsymbol{e}}) + \frac{\|z_1\|}{k_{1e}^2 - z_1^2}\delta_d. \tag{26}$$

Since $V$ is bounded in $\Omega_{\tilde{e}}$, then $\|z_1\|$ is bounded, hence $\dot{V}$ is upper bounded by a bounded negative semidefinite function. Therefore, *uniform local asymptotic stability* of the equilibrium point $\tilde{\boldsymbol{e}} = 0$ is concluded. □

**Corollary 1.** *Notice that for non-persistent disturbances, the proposed scheme exhibits ULES, therefore the scheme offers practical stability in presence of lateral skidding.*

### E. Discussions

As a consequence of $\dot{\alpha}$ in (20), our proposed control law demands a $\mathcal{C}^4$-reference trajectory, requiring only position feedback, not velocities from the WMR.

Given that one of the main limitations of classical symmetric BLF approach is that the domain of attraction is conditioned by the initial conditions of the system. Nonetheless, in the following section, we show that a *Velocity Field* design of desired trajectories overcomes the above limitation, keeping the same simple structure.

## IV. Fuzzy Velocity Field Design of Desired References

We find convenient to consider fuzzy velocity field scheme [10] to design the smooth desired nominal reference. On one hand, the velocity fields arise as *"static description of a motion objective"*, in terms of spatial position, which can be designed without getting into the radial reduction problem [11]. On the other hand, fuzzy logic provides tools on how to embed designer intuition behind the wheel to drive a car: we maneuver faster toward the target when we are far, but steer smoothly when getting closer, within a desired approach angle. Therefore, task description and tuning are simple and intuitive from experience. The velocity field design can mimic such behaviour as a spatial distribution of Cartesian vector fields at any position to describe a smooth instantaneous change of direction, magnitude, and orientation towards a contour of the target. In this way, we can produce vector fields to maneuver smoothly toward target by aligning the WMR to the closest tangent vector of the target contour. This intuition can be encoded by means of the fuzzy sets `CLOSE TO` and `FAR FROM`, while the orientation arises from the $\mathbb{R}^2$ gradient vector field of the desired contour.

Let $f(x,y) = c$ be a level set of the target contour, which is modeled as a smooth, simple oriented, and closed surface. Let $\boldsymbol{r} \in \mathbb{R}^2$ be the vector of an immersed particle position, then $\mathrm{g} = f(\boldsymbol{r}) - c$ describes the signed distance between the current $f(\boldsymbol{r})$ and the desired level set. Let $\mathcal{M}_c := \left\{\boldsymbol{r} \in \mathbb{R}^2 : \mathrm{g}(\boldsymbol{r}) = c\right\}$ be a smooth manifold that defines all positions for a given level set, then, the following conditions are satisfied: 1)$\| \partial \mathrm{g}/\partial \boldsymbol{r} \| \neq 0$, for $c = 0$ and almost any $c \in \mathbb{R}$, 2) $\mathbb{R}^2 = \bigcup_{c\in\mathbb{R}} \mathcal{M}_c$, 3) $\sup\left\{\left\|\dot{\mathcal{V}}\right\|\right\} \in \mathbb{R}$. Hence, taking g as a contouring metric, the velocity field $\mathcal{V}$ arises as, [9],

$$\mathcal{V} := \mu_{\text{close}}(\mathrm{g})\,\boldsymbol{T} + \mu_{\text{far}}(\mathrm{g})\,\boldsymbol{N} \tag{27}$$

where the membership functions satisfy $\mu_{\text{close}} + \mu_{\text{far}} = 1$, and

$$\boldsymbol{e}_{\mathrm{g}} := \frac{\partial \mathrm{g}/\partial \boldsymbol{r}}{\| \partial \mathrm{g}/\partial \boldsymbol{r} \|}, \quad \boldsymbol{N} := -\beta\varphi(\mathrm{g})\,\boldsymbol{e}_{\mathrm{g}} \tag{28}$$

$$\boldsymbol{T} := \pm\alpha \mathrm{R}_{\pi/2}\boldsymbol{e}_{\mathrm{g}}, \quad \mathrm{R}_{\pi/2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{29}$$

where $\varphi(\mathrm{g})$ is designed according to Theorem 2 of [9]; $\alpha, \beta \in \mathbb{R}^+$ are the contour tracking speed, and the convergence rate, respectively. Thus, the existence of integral curves $\gamma(t) = \int_{t_0}^{t} \mathcal{V}(\tau)d\tau$ leads to asymptotic convergence towards the desired contour. Furthermore, for $\mu(\mathrm{g}), \varphi(\mathrm{g}) \in \mathcal{C}^2(\mathbb{R})$, smoothness of $\mathcal{V}, \dot{\mathcal{V}}$ ensures that the vector field is a suitable reference for the chained form system (7).

In contrast to [9], [10], our proposal is based on the Lyapunov direct method, thus pose tracking is guaranteed. Moreover, exponential tracking of orientation error guarantees that any tangent vector of $\mathcal{V}$ will be accessible, and therefore, $\gamma(t)$ will be a valid solution of (4), even in the presence of unmatched disturbances.

### A. Main Result 2

**Proposition 3.** *Consider Proposition 2 and the NH-WMR immersed in the velocity field* (27)*. Then, robot approaches an arbitrarily small vicinity around the contour asymptotically.*

*Proof.* As a direct consequence of Proposition 2, $\tilde{\boldsymbol{e}} \to \Omega_{\tilde{e}}$ (Eq. (23)), hence we obtain the following invariant set for the chained form error $\boldsymbol{e} \to \Omega_e$

$$\Omega_e := \left\{\boldsymbol{e} \in \mathbb{R}^3 : \|e_1\| = 0, \|e_2\| \leq \epsilon_{e_2}, \|e_3\| \leq \epsilon_{z_1}\right\}, \tag{30}$$

where $\epsilon_{e_2} := \delta_d \varepsilon^*_{\nu_{1,d}} / \underline{\varepsilon}_{\nu_{1,d}}$. Now, when $\theta = \theta_d$ as $t \to T$, we can invert the chained form transformation (Eq. (5)) $\boldsymbol{X} = \Gamma(\theta)\,q \Leftrightarrow q_e = \Gamma^{-1}(\theta)\,\boldsymbol{e}$ to obtain the equivalent pose invariant set

$$\Omega_{q_e} := \left\{q_e \in \mathbb{R}^3 : \|x_e\| \leq \varepsilon_x, \|x_e\| \leq \varepsilon_y, \|\theta_e\| = 0\right\}, \tag{31}$$

where $\varepsilon_x := \varepsilon_{e_2} + \varepsilon_{z_1}$, $\varepsilon_y = \varepsilon_{e_2} - \varepsilon_{z_1}$. Finally, from *Theorem 2* of [9] we get that $\int_{t_0}^{t} \mathcal{V} d\tau \to \mathcal{M}_0$ as $t \to \infty$, therefore, $\Omega_{q_e}$ defines a vicinity around the desired contour. □

## V. Simulations

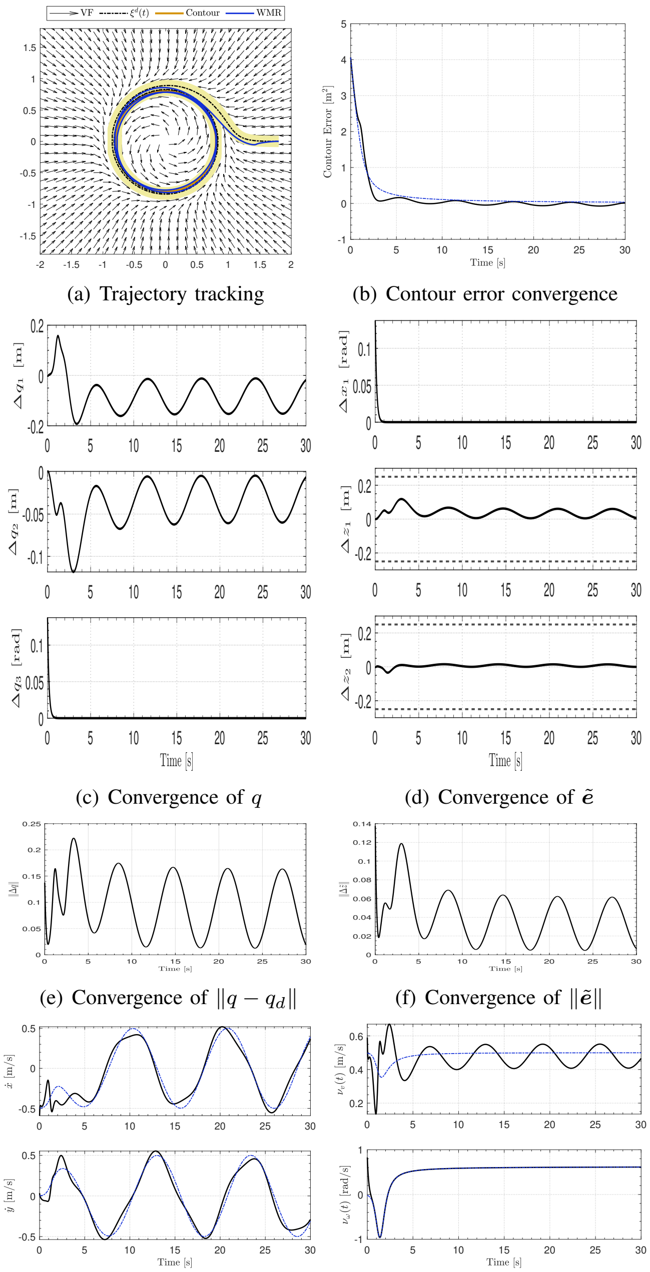
*1) The simulator:* Simulations were programmed in `MATLAB-Simulink-2021b` on a PC equipped with a processor Intel i7-10510U@1.8GHz and 8GB RAM; the *Runge-Kutta4* numerical integrator tuns at a time step of $h = 1$ [ms].

*2) Desired Trajectories:* Consider a simple but representative scenario where the robot task is to maneuver to reach a (target) smooth contour given by $\mathrm{g} = x^2 + y^2 - 0.8^2$, with same value for tracking speed and an approach rate, i.e. $\alpha = \beta = 0.5$ [m/s]. The velocity field is designed in accordance to (27), using the following smooth tuning functions $\mu_{\text{close}} = \tanh^2(\gamma\mathrm{g})$, $\mu_{\text{far}} = \mathrm{sech}^2(\gamma\mathrm{g})$, $\varphi = \tanh(\kappa\mathrm{g})$, for $\gamma = 1.0, \kappa = 100$. Thus, the desired Cartesian trajectory $\xi_d = [x_d, y_d] = \int_{t_0}^{t} \mathcal{V}(\tau)d\tau$, steering angle $\theta_d = \arctan \mathcal{V}_y/\mathcal{V}_x, v_d = \|\mathcal{V}\|, \omega_d = \left(\dot{\mathcal{V}}_y\mathcal{V}_x - \mathcal{V}_y\dot{\mathcal{V}}_x\right)/\|\mathcal{V}\|^2$.

*3) Initial conditions, control gains, barrier width, and disturbance:* The robot initial pose is $q(t_0) = (1.8\text{ [m]}, 0\text{ [m]}, 3\text{ [rad]})$. To illustrate the effectiveness of the proposed approach, a severe time varying unmatched disturbance is introduced: $d(t) = 0.1(1 + \sin t)$ [m/s]. The barrier width is $k_{1e} = k_{2e} = 0.25$ [m], and $k_0 = 6.15, k_1 = 4.5, k_2 = 17.25$ as the controller gains.

*4) Results:* A smooth maneuvering through the tracking of the integral curve of the velocity field is shown in Figs. 2(a) and 2(b). Smooth convergence and tracking are obtained for pose errors, with error norms varying in accordance to control activity, see Figs. 2(c) and 2(e), and Fig. 2(h).

We observe in Fig. 2(a) that the mobile robot drifts away from the desired flow line during the contour error transient, producing an overshoot for Cartesian coordinates $(x, y)$ at $t = 1.5,\ 3.5$[s], however the robot remains within the barriers. This effect is expected due to the unmatched disturbance $d$ and the high steering rate needed to converge to the contour from initial configuration, see Fig. 2(h). While the position




(a) Trajectory tracking (b) Contour error convergence

(c) Convergence of $q$ (d) Convergence of $\tilde{e}$

(e) Convergence of $\|q - q_d\|$ (f) Convergence of $\|\tilde{e}\|$

(g) Velocity field tracking (h) Control signals

Fig. 2. The disturbed NH-WMR tracking a fuzzy velocity field.

overshoot can be bounded setting a higher $k_1$, or enlarging the $k_{1e}$ constraint, a smoother steering $\omega_d$ depends upon tuning the velocity field parameters $\kappa, \beta$. Despite the overshoot, the forward velocity control signal, shown in Fig. 2(h), evidence that the robot effectively retains its position within the barriers. Smooth asymptotic behaviour toward an arbitrary small set around the origin is observed, see Figs. 2(d) to 2(f), along the velocity field, as shown in Figs. 2(b) and 2(g). In addition, norms of $q$ and errors are shown in Figs. 2(c) and 2(f), providing a clear insight of Proposition 3, that is, the tracking performance of the proposed controller. Notice that most of the pose error norm arises from the $x$-coordinate error, see Fig. 2(c). This is consistent with (31) as $\epsilon_x > \epsilon_y$, to keep a proper tracking on $(\theta, y)$ coordinates.

## VI. Conclusion

A BLF control scheme has been proposed for a nonholonomic mobile robot subject to unmatched disturbances to track a velocity field that maneuvers smoothly toward a target. Fixed barrier width enforces boundary of the WMR trajectories to converge to a small neighborhood of target. Numerical study suggests that the proposed velocity field approach is suitable to steer a physical wheeled robot in a realistic setting of unmatched disturbances arising from considering some relevant effects of tire deformation, essentially, the effect that pressure contact point moves around the surface contact area. Simulations show that our proposed approach *"compromises" the $x$-coordinate tracking*, as consequence of the backstepping procedure, while the dependence $\delta_d$ on $\epsilon_x$ makes it difficult to reduce the attraction region. Due to the latter, the steering velocity remains unchanged, while the forward velocity stabilizes the position subsystem; hence, the proposed approach deals with the unmatched disturbance by changing the forward rate. As future work, skidding and slipping at a dynamical level is under consideration, and experiments are under way to validate its real time performance, using a motion capture system to oversee such unmatched disturbances.

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