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Adaptive Tracking Control of an Uncertain Duffing-Holmes System

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***Abstract*—This paper presents a backstepping-based adaptive control design for a Duffing-Holmes system with an uncertain parameter. The proposed controller uses an adaptation rule to estimate this parameter and to follow a reference model. The contraction theory is used to analyze the convergence of the tracking and parameter estimation errors. The performance of the proposed controller is validated through simulation studies.**

***Index Terms*—Backstepping design, contraction analysis, duffing oscillator, parameter identification.**

I. INTRODUCTION

The chaotic theory is a popular topic within the research area of nonlinear dynamics. For certain applications like chaotization, a chaotic behavior is desired, but it may cause undesired complex behavior and needs to be controlled. For that reason, the control and synchronization of chaotic systems have been of interest to many researchers for decades [1]–[3].

It is known that the dynamics of chaotic systems depend on both their parameters and initial conditions [4]. The proper knowledge of the system parameters can help to improve the overall control performance. Most of the reported works assume that the parameters of chaotic systems are known in advance, which is not true in a real-world scenario due to their complex behavior. Thus, a central problem in the control design is exploring how to obtain these unknown parameters. They can be obtained either by using parametric identification techniques [5], or by non-parametric techniques such as the neural networks [6], fuzzy logic [7], and chaotic swarm methods [8]. Meanwhile, a sliding mode controller is a potential candidate for handling system uncertainties [9]. However, the resulting control signal is discontinuous, limiting its practical implementation on certain mechanical systems. On the other hand, an adaptive controller can be used to adapt for the unknown parameters [10]. Most of these adaptation laws are derived using the traditional Lyapunov theory; nonetheless, the proper selection of a suitable Lyapunov function remains as a design challenge [11].

This paper presents a new adaptive tracking control design for a Duffing-Holmes system. The system parameter is assumed to be uncertain and the adaptive control is applied to identify this parameter for tracking a reference model. The contraction theory [12] is employed to analyze the stability of the controller. Compared to the traditional Lyapunov stability analysis, that studies the convergence of the system trajectories with respect to an equilibrium point, the contraction theory examines incremental stability, where the convergence of the system trajectories to a given trajectory is analyzed [13]. Moreover, the convergence of the parameter estimation error to zero is theoretically demonstrated by showing that it relies on a persistency of excitation condition.

The paper is organized as follows. Section II gives some useful results from Contraction theory. Section III discusses the adaptive tracking control design for Duffing-Holmes systems. Simulation results for the proposed tracking controller are presented in section IV. Finally, Section V provides the conclusions of this paper.

II. CONTRACTION ANALYSIS

The contraction analysis is a form of incremental stability, where the convergence of the system trajectories is studied using differential geometrical analysis. Let us consider the following system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{1}$$

where $\mathbf{f} : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ is a smooth nonlinear function vector, $\mathbf{x} \in \mathbb{R}^n$ is the state vector, and $t \in \mathbb{R}$ is the time. The following two Definitions and Remark are presented in [12].

Definition 1. *For the system (1), a region of the state-space is called a contraction region if the Jacobian $\mathbf{J} = \partial \mathbf{f} / \partial \mathbf{x}$ is uniformly negative-definite (UND) in this region.*

Definition 2. *A state-space region of the system (1) is called a semi-contraction region if $\mathbf{J}$ is uniformly negative semi-definite in this region.*

Remark 1. *In a contraction region, the system trajectories forget their initial condition and converge to zero at an exponential rate of $\lambda_{\max}$, which is the largest eigenvalue of the symmetric part of the Jacobian $\frac{1}{2}\left(\mathbf{J} + \mathbf{J}^{\mathrm{T}}\right)$, whereas in a semi-contracting region they converge to zero asymptotically.*

Let us consider that the system (1) is controlled using a reference model-based adaptive tracking controller and the corresponding closed-loop dynamics is represented by

$$\dot{\mathbf{z}} = \mathbf{g}(\mathbf{z}, t) - \mathbf{Y}(\mathbf{x}, t)\boldsymbol{\theta} \tag{2}$$

where $\mathbf{g}(\mathbf{z}, t) \in \mathbb{R}^n$ is a bounded and known function vector, $\mathbf{Y}(\mathbf{x}, t) \in \mathbb{R}^{n\times m}$ is a bounded regressor matrix, whose entries are exactly known signals that could be a linear or nonlinear function of $\mathbf{x}$, and $\boldsymbol{\theta} \in \mathbb{R}^m$ is a constant and unknown parameter vector. Moreover, the vector $\mathbf{z} \in \mathbb{R}^n$ depends on the tracking error $\mathbf{x} - \mathbf{x}_d$, where $\mathbf{x}_d$ is a smooth bounded desired trajectory obtained using a reference model of (1).

Assume that the product $\mathbf{Y}(\mathbf{x}, t)\boldsymbol{\theta}$ is a linear parameterization, then (2) can be rewritten as

$$\dot{\mathbf{z}} = \mathbf{g}(\mathbf{z}, t) - \mathbf{Y}(\mathbf{x}, t)\widehat{\boldsymbol{\theta}} + \mathbf{Y}(\mathbf{x}, t)\widetilde{\boldsymbol{\theta}} \tag{3}$$

where $\widehat{\boldsymbol{\theta}}$ is the estimate of $\boldsymbol{\theta}$ and $\widetilde{\boldsymbol{\theta}} = \widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}$ is the parameter estimation error vector. Then $\widehat{\boldsymbol{\theta}}$ can be estimated using the following adaptation law:

$$\dot{\widehat{\boldsymbol{\theta}}} = \dot{\widetilde{\boldsymbol{\theta}}} = -\mathbf{Y}^{\mathrm{T}}(\mathbf{x}, t)\mathbf{z}. \tag{4}$$

Finally, computing the Jacobian of the closed-loop system (3) and the adaptation law (4) with respect to the state vector $[\mathbf{z}, \widetilde{\boldsymbol{\theta}}]^{\mathrm{T}}$, leads to the following virtual dynamics

$$\frac{d}{dt}\begin{bmatrix} \delta\mathbf{z} \\ \delta\widetilde{\boldsymbol{\theta}} \end{bmatrix} = \begin{bmatrix} \mathbf{J}(\mathbf{z}, t) & \mathbf{Y}(\mathbf{x}, t) \\ -\mathbf{Y}^{\mathrm{T}}(\mathbf{x}, t) & \mathbf{0} \end{bmatrix}\begin{bmatrix} \delta\mathbf{z} \\ \delta\widetilde{\boldsymbol{\theta}} \end{bmatrix} \tag{5}$$

where $\mathbf{J}(\mathbf{z}, t) = \partial\mathbf{g}(\mathbf{z}, t)/\partial\mathbf{z}$. The convergence property of (5) is described using the following lemma.

Lemma 1 ([14]). *Assume that $\mathbf{J}(\mathbf{z}, t)$ is UND, hence (2) is contracting in $\mathbf{z}$. Then, the closed-loop system (5) is semi-contracting, which means that $\mathbf{z}$ asymptotically converges to zero and the parameter estimation error $\widetilde{\boldsymbol{\theta}}$, resulting from the adaptation law (4), remains bounded.*

III. Adaptive Tracking Control Design

This section describes an adaptive tracking control design based on a backstepping technique for an uncertain Duffing-Holmes oscillator. The behavior of this system depends on its initial condition. The control design goal is to obtain a closed-loop contracting system; hence the system forgets its initial condition and tracks the desired trajectory by simultaneously identifying the unknown system parameter.

Consider the Duffing-Holmes system described by the following equation

$$\ddot{x} + 2\zeta\dot{x} - x + x^3 = u \tag{6}$$

where ζ is the unknown damping ratio, u is the control input, and $\ddot{x}, \dot{x}$, and x are the acceleration, velocity, and position, respectively. For $u = 0$, the above system has two stable equilibrium points at $(x^*, \dot{x}^*) = (\pm 1, 0)$. This bistability is due to the presence of the nonlinear restoring force $f_r = -x + x^3$, which is constituted by a negative linear and a positive nonlinear stiffness term. The objective is to design a tracking controller such that $x \to x_d$, where x_d is a smooth bounded desired trajectory.

By letting $x_1 = x$ and $x_2 = \dot{x}$, the system (6) can be represented in the following state-space form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2\zeta x_2 + x_1 - x_1^3 + u. \end{aligned} \tag{7}$$

Let us define the following new state variables

$$z_1 = x_1 - x_d \tag{8}$$
$$z_2 = x_2 - \alpha \tag{9}$$

where α is a virtual control input that will be chosen later.

The dynamics of z_1 is calculated as

$$\dot{z}_1 = \dot{x}_1 - \dot{x}_d = z_2 + \alpha - \dot{x}_d. \tag{10}$$

Choosing α as follows

$$\alpha = \dot{x}_d - k_1 z_1 \tag{11}$$

where k_1 is the control gain, then (10) becomes

$$\dot{z}_1 = -k_1 z_1 + z_2. \tag{12}$$

The dynamics of z_2 is $(\dot{x}_2 - \dot{\alpha})$ and is given by

$$\dot{z}_2 = z_1 + x_d - (z_1 + x_d)^3 - 2\zeta(z_2 + \alpha) - \dot{\alpha} + u \tag{13}$$

where $\dot{\alpha} = \ddot{x}_d - k_1\dot{z}_1$. Define the following proposed control law

$$u = -2z_1 - x_d + (z_1 + x_d)^3 - k_2 z_2 + 2\widehat{\zeta}(z_2 + \alpha) + \dot{\alpha} \tag{14}$$

where $\widehat{\zeta}$ is an estimate of ζ. Substituting this u into (13) produces

$$\dot{z}_2 = -z_1 - k_2 z_2 + 2x_2\widetilde{\zeta} \tag{15}$$

The term $\widetilde{\zeta} = \widehat{\zeta} - \zeta$ is the parameter estimation error, and ζ is estimated using the following adaptation law

$$\dot{\widehat{\zeta}} = -2x_2 z_2. \tag{16}$$

Theorem 1. *Consider the dynamic system (7) with the unknown parameter ζ, that uses the control law (14) and the parameter updating law (16). By choosing positive control gains k_1 and k_2, then x asymptotically converges to the desired trajectory x_d and the parameter estimation error $\widetilde{\zeta}$ remains bounded.*

Proof. The virtual dynamics of the closed-loop system consisting of (12), (15), and (16) is expressed as

$$\frac{d}{dt}\begin{bmatrix} \delta z_1 \\ \delta z_2 \\ \delta\widetilde{\zeta} \end{bmatrix} = \begin{bmatrix} -k_1 & 1 & 0 \\ -1 & -k_2 & 2x_2 \\ 0 & -2x_2 & 0 \end{bmatrix}\begin{bmatrix} \delta z_1 \\ \delta z_2 \\ \delta\widetilde{\zeta} \end{bmatrix} \tag{17}$$

which has the same structure of (5) by defining $\theta = \zeta$,

$$\mathbf{J}(\mathbf{z}) = \begin{bmatrix} -k_1 & 1 \\ -1 & -k_2 \end{bmatrix}, \text{ and } \mathbf{Y}(\mathbf{x}, t) = \begin{bmatrix} 0 \\ 2x_2 \end{bmatrix}$$

where $\mathbf{z} = [z_1, z_2]^{\mathrm{T}}$ and $\mathbf{x} = [x_1, x_2]^{\mathrm{T}}$. Note that $\mathbf{J}(\mathbf{z})$ is UND since $\frac{1}{2}\left(\mathbf{J} + \mathbf{J}^{\mathrm{T}}\right) = \mathrm{diag}\{-k_1, -k_2\}$; then by using Lemma 1, it can be concluded that the closed-loop system (17) is semi-contracting, implying that $\delta\mathbf{z}$ goes to zero asymptotically and $\delta\widetilde{\zeta}$ remains bounded. Now, using (8), it can be concluded that the asymptotic convergence of δz_1 to zero also means that the system state x_1 asymptotically converges to the desired trajectory x_d. In addition, note that $\delta z_2 = \delta\dot{z}_1 + k_1\delta z_1$, then $\delta\dot{z}_1$ also converges to zero asymptotically. Then by using (10), we can conclude that x_2 asymptotically converges to $\dot{x}_d$. Moreover, the parameter estimation error $\widetilde{\zeta}$ remains bounded. □

Remark 2. *In the absence of parametric uncertainty, the application of the controller (14) with $\widehat{\zeta} = \zeta$ produces a contracting closed-loop system. Hence, the system states x_1 and x_2 converge to the desired trajectories x_d and $\dot{x}_d$, respectively, at an exponential rate of $\lambda_{\max} = \max\{-k_1, -k_2\}$. Furthermore, a faster convergence rate can be achieved by using larger controller gains k_1 and k_2.*

The convergence of the parameter estimation error $\widetilde{\zeta}$ is analyzed in the next paragraphs.

Definition 3 ([15]). *A vector $\boldsymbol{\psi} \in \mathbb{R}^m$ is persistenly exciting (PE) if it satisfies*

$$\frac{1}{T_0}\int_t^{t+T_0} \boldsymbol{\psi}(\tau)\boldsymbol{\psi}^{\mathrm{T}}(\tau)d\tau \geq \alpha_0\mathbf{I}, \quad \forall t \geq 0 \tag{18}$$

for some positive constants α_0 and $T_0 > 0$.

Definition 4 ([15]). *A signal $v \in \mathbb{R}$ is called sufficiently rich of order m, if the support of its spectral measure contains at least m points.*

Theorem 2. *Let the Laplace transform of ψ given by*

$$\mathcal{L}[\boldsymbol{\psi}] = \mathbf{H}(s)\mathcal{L}[v] \tag{19}$$

and assume that $\mathbf{H}(j\omega_1)$, ...,$\mathbf{H}(j\omega_m)$ are linearly independent on the complex space $\forall\omega_1, \ldots .\omega_m \in \mathbb{R}$, such that $\omega_i \neq \omega_j$ for $i \neq j$. Then, $\boldsymbol{\psi}$ is PE if and only if v is sufficient rich of order m.

Proof. See [15]. □

Proposition 1. *Assume that the desired trajectory x_d for system (6) is the output of the following reference model*

$$\ddot{x}_d + 2\zeta_r\dot{x}_d - x_d + x_d^3 = F \tag{20}$$

$$F = a\sin(\omega t) \tag{21}$$

where ζ_r is the damping ratio of the reference model, and F is an external vibration force that produces a limit-cycle in the model (20), i.e., the response x_d converges in steady-state to a periodic orbit. Then, the estimate $\widehat{\zeta}$ of the adaptation law (16) converges to the parameter ζ of the Duffing-Holmes system in (6).

Proof. We have that $z_i \to 0$, $i = 1, 2$ and $x_2 \to \dot{x}_d$. Moreover, $\ddot{z}_2$ is given by

$$\ddot{z}_2 = -\dot{z}_1 - k_2\dot{z}_2 + 2\left(\dot{x}_2\widetilde{\zeta} + x_2\dot{\widetilde{\zeta}}\right). \tag{22}$$

Note that $\ddot{z}_2$ is bounded because all the terms of the right hand side of (22) are bounded; therefore, signal $\dot{z}_2$ is uniformly continuous. Then, according to the Barbalat's lemma [15], the signal $\dot{z}_2 \to 0$ since $z_2 \to 0$ and $\dot{z}_2$ is uniformly continuous. Thus, for a sufficient long time t, (15) is converted to

$$\dot{x}_d\widetilde{\zeta} = 0. \tag{23}$$

Multiplying (23) by $\dot{x}_d/T_0$ and integrating the resulting expression over a period T_0 yields

$$\left[\frac{1}{T_0}\int_t^{t+T_0} \dot{x}_d^2(\tau)d\tau\right]\widetilde{\zeta} = 0. \tag{24}$$

If $\dot{x}_d$ is PE, then the only solution of (24) is $\widetilde{\zeta} = 0$. In steady-state, the signals x_d and $\dot{x}_d$ of the reference model (20) are almost sinusoidal with the same frequency ω of F [16]. Moreover, these signals contain subharmonic and superharmonic resonances of ω due to the nonlinear stiffness term x_d^3 of the reference model [17]. Thus, the spectrum of x_d and $\dot{x}_d$ contains more than 2 points, and according to Definition 4, the signal $\dot{x}_d$ is sufficiently rich of order $m = 1, 2$ or more. Select $H(s) = 1$ and $v = \psi = \dot{x}_d$ in (19); then $H(j\omega_1)$ is linearly independent for $\omega_1 \in \mathbb{R}$. Since $\dot{x}_d$ is sufficiently rich of order $m = 1$, Theorem 2 guarantees that $\dot{x}_d$ is PE and $\widehat{\zeta}$ converges to ζ. □

IV. Simulation Results

In this section, simulation studies are presented to evaluate the performance of the proposed controller (14) as well as the parameter updating law (16). The simulation studies are performed in the MATLAB/Simulink software at a sampling period of 1 ms, using the numerical integration solver `ode45` based on the Dormand-Prince method.

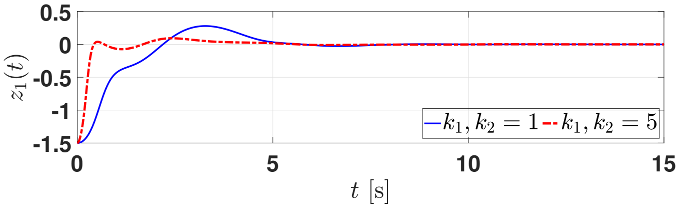
The Duffing-Holmes system (6) with $\zeta = 0.01$ is controlled using the controller (14) with gains $k_1 = k_2 = 1$ along with the parameter adaptation law (16). The initial conditions are chosen to be $\mathbf{x}(0) = [-1, 1]^{\mathrm{T}}$ and $\widehat{\zeta}(0) = 0$, and the desired trajectory is generated using the reference model given in (20) with $\zeta_r = 0.01$, $F = 0.08\sin(0.8t)$, and $\mathbf{x}_d(0) = [0.5, 1]^{\mathrm{T}}$. The results are shown in Fig. 1, which indicate that the controller achieves a good tracking performance. Remarkably, Fig. 1(b) shows that the estimated parameter converges to the actual parameter around 10 s. The simulation is repeated with the controller gains $k_1 = k_2 = 5$, that produce a faster tracking performance than with the gains $k_1 = k_2 = 1$.

Finally, a simulation study is performed by choosing a different damping parameter for the reference model (20), given by $\zeta_r = 0.03$. The initial conditions are set as $\mathbf{x}(0) = [-1, 1]^{\mathrm{T}}$, $\widehat{\zeta}(0) = -0.1$, and $\mathbf{x}_d(0) = [0.5, 1]^{\mathrm{T}}$. The controller (14) with gains $k_1 = k_2 = 1$ is used, and the corresponding tracking performance is shown in Fig. 2. Despite the changes in the parameters ζ_r and $\widehat{\zeta}(0)$, the convergence time of the parameter estimate to the nominal parameter remains the same as the previous case.

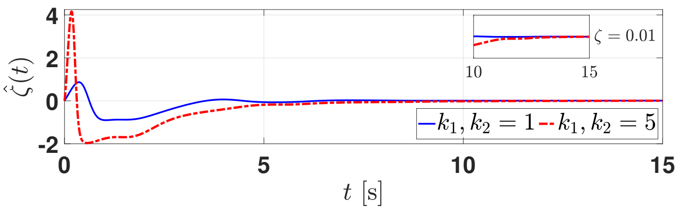
From these results, we can see that the estimated parameter $\widehat{\zeta}$ converges to its actual value ζ irrespective of its initial condition; moreover, the tracking controller drives the system to the desired trajectory.

V. Conclusions

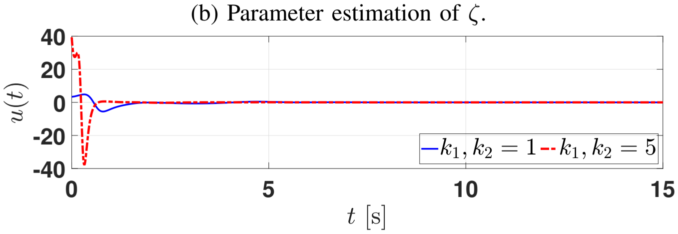
This paper proposed a novel adaptive tracking controller for Duffing-Holmes systems with parametric uncertainty. The controller uses a parameter identification method to improve



(a) Position tracking error z_1.

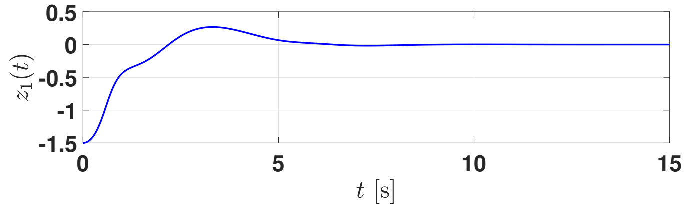


(b) Parameter estimation of ζ.

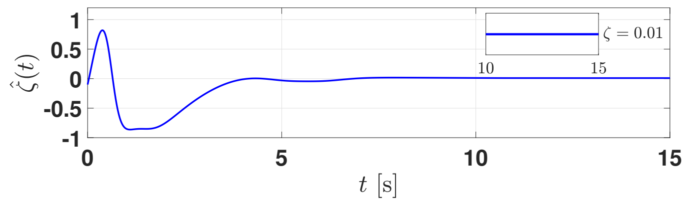


(c) Control signal u in (14).

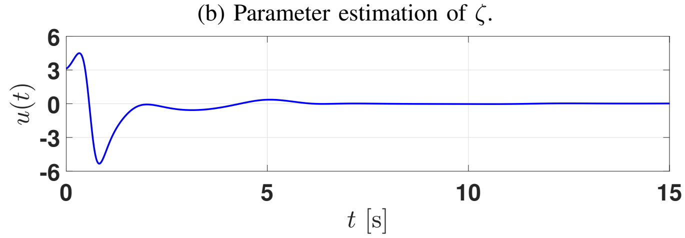
Fig. 1: Tracking control of Duffing-Holmes system (6) with different control gains.



(a) Position tracking error z_1.



(b) Parameter estimation of ζ.



(c) Control signal u in (14).

Fig. 2: Tracking control of Duffing-Holmes system (6) with $\zeta \neq \zeta_r$.

its tracking performance. The simulation results show that the parameter identification and tracking performance are achieved successfully for different parameter and control gain settings. Moreover, the proposed adaptive controller is insensitive to the initial conditions.

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