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A Systematic Method for Backstepping via Linear Matrix Inequalities

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***Abstract*—In this paper, a systematic method for implementing backstepping via linear matrix inequalities is developed. It is based on using the direct Lyapunov method at each step of the backstepping procedure in order to find a control law as a nonlinear convex sum of gains. These gains are obtained via commercially available software for solving linear matrix inequalities, which guarantees numerical readiness of the proposal. Moreover, dealing with MIMO subsystems at each step can be done without further adjustments. Examples of academic and practical interest illustrate the advantages of the novel methodology over former approaches.**

***Index Terms*—Backstepping, Linear Matrix Inequality, Convex Embedding.**

I. Introduction

Among nonlinear control techniques, backstepping stands as one of the most genuinely exploiting the nonlinear nature of the system, provided it is amenable to a cascade structure and fulfills certain conditions on controllability, smoothness, and size [1]. In contrast with feedback linearization [2] and ad-hoc Lyapunov-based approaches [3], backstepping does not necessarily cancel out the system nonlinearities nor is attached to a fixed Lyapunov structure or control law form, i.e., it takes advantage of the system cascade structure while leaving many free options for handling the particular design on every step of the procedure. The cascade form is not necessarily a limitation: it can be obtained via appropriate diffeomorphisms that transform the nonlinear system into its normal form [4].

Despite its usefulness, backstepping is not without disadvantages: as many other nonlinear control techniques, it relies on the designer's ability to provide a Lyapunov function or a control law at each step of the procedure. Indeed, these steps are usually analytical and may require a certain dose of ingenuity to propose a Lyapunov function candidate that actually works and a control law able to handle the nonlinear nature of certain parts, especially those with multiple-input multiple-output (MIMO) characteristics [5]. Due to these reasons, generalizations of the backstepping technique may involve only-scalar inputs at each time, handling norm-based inequalities for establishing negative-definiteness of expressions, and invertibility of any factor multiplying the virtual inputs at each step [6].

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On the other hand, in the last two decades nonlinear control based on convex modelling and convex optimization has drawn the attention of many researchers [7]. Originally appeared in the context of linear parameter-varying (LPV) [8] and Takagi-Sugeno (TS) systems [9], the lure of this approach has been both its systematicness and numerical implementability, which compare advantageously with other nonlinear techniques. The standard methodology is based on a convex rewriting of the nonlinear system [10], a nonlinear control law as a convex sum of gains known as parallel distributed compensation (PDC) [11], and the direct Lyapunov method for translating the conditions for asymptotic stability of the origin into linear matrix inequalities (LMIs) [12], which are efficiently solved via commercially available software such as SeDuMi [13] or LMI Toolbox [14]. The designer is not asked to provide a different Lyapunov function every time a new system is considered: it suffices to provide the convex model vertices in order to readily test the LMI conditions; if feasible, gains in the PDC control law are immediately obtained and asymptotic stability of the closed-loop system guaranteed for the outermost Lyapunov level where convexity of the rewritten model is guaranteed [15].

The nice properties of nonlinear control based on convex models and LMIs have been increasingly sought in many other areas such as sliding modes [16], output regulation [17], fractional-order [18], singular systems [19], neural identification [20], computed torque [21], etc. In the same line, this paper proposes a way to merge LMI-based convex control with backstepping, as to improve the latter with the numerical efficiency and systematicness of the former. Moreover, the combination allows for MIMO subsystems to be easily considered at each step of backstepping as such characteristic makes no difference in the LMI-based convex control approach. Since the latter is based on the direct Lyapunov method, it directly fulfills the backstepping requirement of constructing a nested Lyapunov function by means of those obtained step by step.

The paper is organized as follows: section II provides the necessary background on backstepping, convex modelling, and LMI-based control; section III develops the main results, namely, a systematic methodology for LMI-based implementation of backstepping; section IV illustrates the effectiveness of the proposal as well as its advantages over former approaches, by means of academic and practical examples; finally, conclusions and future work are discussed in section V.

II. Preliminaries

This section consists in two parts: the first one concerns the basics on backstepping; the second one presents convex modelling and LMI-based control via the direct Lyapunov method. Both approaches will be merged in the next section.

A. Backstepping

Backstepping is a well-known nonlinear control technique that exploits the cascade connection in nonlinear systems of the form

$$\dot{\eta}(t) = f_1(\eta) + g_1(\eta)\xi(t), \quad (1)$$
$$\dot{\xi}(t) = f_2(\eta,\xi) + g_2(\eta,\xi)u(t), \quad (2)$$

where the state vector is split in $\eta \in \mathbb{R}^n$ and $\xi \in \mathbb{R}$; the control input is $u(t) \in \mathbb{R}$; $f_i(\cdot)$, $g_i(\cdot)$, $i \in \{1,2\}$, are nonlinear functions of adequate dimensions that are sufficiently smooth in a domain $\mathscr{D}$ that contains the origin $(\eta,\xi) = (0,0)$ with $f_1(0) = 0$.

Backstepping is performed following the next steps [1]:

1) Consider ξ the control input of the first subsystem (1) and design a feedback control law $\xi(t) = \phi_1(\eta)$, $\phi_1(0) = 0$ guaranteeing that the origin $\eta = 0$ of the closed-loop subsystem is asymptotically stable.
2) Rewrite (1)-(2) as

$$\dot{\eta}(t) = f_1(\eta) + g_1(\eta)\phi_1(\eta) + g_1(\eta)\zeta(t), \quad (3)$$
$$\dot{\zeta}(t) = f_2(\eta,\xi) - \frac{\partial \phi_1}{\partial \eta}\left(f_1(\eta) + g_1(\eta)\phi_1(\eta) + g_1(\eta)\zeta\right) + g_2(\eta,\xi)u(t), \quad (4)$$

where $\zeta = \xi - \phi_1(\eta)$.
3) Design a feedback control law $u(t) = \phi_2(\eta,\xi,\zeta)$, $\phi_2(0) = 0$ guaranteeing that the origin $(\eta,\zeta) = (0,0)$ of the rewritten closed-loop system (3)-(4) is asymptotically stable.

Normally, asymptotic stability of the origin is guaranteed via the direct Lyapunov method. Therefore, it is quite common to find a Lyapunov function $V_1(\eta)$ for the first step guaranteeing that

$$\frac{\partial V_1}{\partial \eta}\left(f_1(\eta) + g_1(\eta)\phi(\eta)\right) \leq -W(\eta),$$

for a positive-definite function $W(\eta)$. Similarly, a Lyapunov function $V_2(\eta,\xi)$ has to be found for the rewritten closed-loop system (3)-(4); an usual option is $V_2(\eta,\xi) = V_1(\eta) + 1/2\zeta^2$. This option may help the designer to find $u(t)$ such that $\dot{V}_2 < 0$ in $\mathscr{D}$ since $u(t)$ is a scalar input enabled to cancel all the nonlinearities.

Naturally, if a system can be put in a series of cascade connections such as the one presented above, backstepping can be recursively applied [5], [6].

B. Convex modelling and LMI-based control

LMI-based nonlinear control usually begins with a convex embedding of the nonlinear plant model [10]; this convex form helps obtaining LMIs for controller design from the application of the direct Lyapunov method [9]. To see this, consider a nonlinear system of the form

$$\dot{x}(t) = f(x) + g(x)u(t), \quad (5)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input vector, $f(\cdot)$ and $g(\cdot)$ are sufficiently smooth functions of adequate dimensions; the methodology consists in performing the following steps:

1) Rewrite (5) as

$$\dot{x}(t) = A(x)x(t) + B(x)u(t), \quad (6)$$

with $A(x) \in \mathbb{R}^{n\times n}$ can be factorized in a variety of ways, and $B(x) = g(x) \in \mathbb{R}^{n\times m}$.
2) Let $z_j(x) \in [z_j^0, z_j^1]$, $j \in \{1,2,\ldots,p\}$ be the different non-constant terms in $A(x)$ and $B(x)$; rewrite all of them as a convex sum of their bounds (which might be induced by a compact set of interest $\Omega \subset \mathbb{R}^n$, $0 \in \Omega$, $x \in \Omega \subset \mathbb{R}^n$), i.e.:

$$z_j(x) = \sum_{i_j=0}^{1} w_{i_j}^j(x) z_j^{i_j}, \quad \text{with}$$
$$w_0^j(x) \equiv \frac{z_j^1 - z_j(x)}{z_j^1 - z_j^0},\ w_1^j(x) = 1 - w_0^j(x).$$

3) Based on the previous expressions, rewrite (6) as

$$\dot{x}(t) = \sum_{\mathbf{i}\in\mathbb{B}^p} \mathbf{w}_{\mathbf{i}}(x)\left(A_{\mathbf{i}}x(t) + B_{\mathbf{i}}u(t)\right), \quad (7)$$

where $\mathbf{i} = (i_1, i_2, \ldots, i_p)$, $\mathbb{B} = \{0,1\}$, $\mathbf{w}_{\mathbf{i}}(x) = w_{i_1}^1(x)w_{i_2}^2(x)\cdots w_{i_p}^p(x)$, $A_{\mathbf{i}} = A(x)|_{\mathbf{w_i}=1}$ and $B_{\mathbf{i}} = B(x)|_{\mathbf{w_i}=1}$. Importantly, this model is algebraically equivalent to the original one.
4) Consider a PDC control law

$$u(t) = \sum_{\mathbf{j}\in\mathbb{B}^p} \mathbf{w}_{\mathbf{j}}(x)F_{\mathbf{j}}x(t), \quad (8)$$

with gains $F_{\mathbf{j}}$, $\mathbf{j} \in \mathbb{B}^p$, to be found later, which produces the closed-loop system

$$\dot{x}(t) = \sum_{\mathbf{i}\in\mathbb{B}^p}\sum_{\mathbf{j}\in\mathbb{B}^p} \mathbf{w}_{\mathbf{i}}(x)\mathbf{w}_{\mathbf{j}}(x)\left(A_{\mathbf{i}} + B_{\mathbf{i}}F_{\mathbf{j}}\right)x(t). \quad (9)$$

5) By means of the quadratic Lyapunov function candidate $V(x) = x^T Px$, $P = P^T > 0$, it can be established that the origin $x = 0$ of the closed-loop system (9) is asymptotically stable if there exist matrices $X \in \mathbb{R}^{n\times n}$ and $M_{\mathbf{j}} \in \mathbb{R}^{m\times n}$, $\mathbf{j} \in \mathbb{B}^p$, such that the LMIs

$$X = X^T > 0, \quad (10)$$
$$\text{He}\left(A_{\mathbf{i}}X + B_{\mathbf{i}}M_{\mathbf{j}}\right) < 0, \quad (11)$$

are feasible $\forall \mathbf{i}, \mathbf{j} \in \mathbb{B}^p$, where $\text{He}(Y) = Y + Y^T$. In that case, $P = X^{-1}$ and $F_{\mathbf{j}} = M_{\mathbf{j}}X^{-1}$. The proof is based on the convexity of functions $\mathbf{w_i}(x)$ in Ω; it can be found in a variety of works, e.g., [9] and [15].

III. Main Results

The following proposal for merging backstepping with LMI-based convex control involves performing the following at each step of the backstepping procedure:

1) rewrite the subsystem under consideration in a convex form,
2) apply a PDC control law for the corresponding input, whether final (real) or fictitious (intermediate),
3) use a quadratic Lyapunov function candidate for determining the gains of the PDC control law via LMIs,
4) construct the final Lyapunov function and the final control law by means of recursion and nesting.

Let us detail the aforementioned steps. We begin with a slight generalization of system (1)-(2) in section II-A, namely,

$$\dot{\eta}(t) = f_1(\eta) + g_1(\eta)\xi(t), \tag{12}$$

$$\dot{\xi}(t) = f_2(\eta,\xi) + g_2(\eta,\xi)u(t), \tag{13}$$

where the state vector is split in $\eta \in \mathbb{R}^{n_1}$ and $\xi \in \mathbb{R}^{n_2}$; the control input is $u(t) \in \mathbb{R}^m$; $f_i(\cdot)$, $g_i(\cdot)$, $i \in \{1,2\}$, are nonlinear functions of adequate dimensions that are sufficiently smooth in a domain $\mathscr{D}$ that contains the origin $(\eta,\xi) = (0,0)$ with $f_1(0) = 0$. Note that fictitious or real inputs, ξ or u, respectively, are no longer required to be scalars; the cascade connection, nevertheless, is preserved.

By means of the convex embedding of expressions, described in II-B, it is clear that (12) can be written as

$$\begin{aligned}\dot{\eta}(t) &= A^1(\eta)\eta(t) + B^1(\eta)\xi(t) \\ &= \sum_{\mathbf{i}\in\mathbb{B}^{p_1}} \mathbf{w}^1_{\mathbf{i}}(\eta)\left(A^1_{\mathbf{i}}\eta(t) + B^1_{\mathbf{i}}\xi(t)\right),\end{aligned} \tag{14}$$

where $A^1(\eta)\eta = f_1(\eta)$, $B^1(\eta) = g_1(\eta)$, p_1 is the number of different non-constant terms in $A^1(\eta)$ and $B^1(\eta)$, $\mathbf{w}^1_{\mathbf{i}}(\eta)$ are constructed from these terms according to the convex-embedding methodology in II-B, taking into account that they are bounded if η belongs to a compact set $\Omega_1 \subset \mathbb{R}^{n_1}$ containing $\eta = 0$, $A^1_{\mathbf{i}} = A^1(\eta)|_{\mathbf{w}^1_{\mathbf{i}}=1}$, and $B^1_{\mathbf{i}} = B^1(\eta)|_{\mathbf{w}^1_{\mathbf{i}}=1}$, $\mathbf{i} \in \mathbb{B}^{p_1}$.

Consider a PDC control law of the form

$$\xi(t) = F^1(\eta)\eta(t) \equiv \sum_{\mathbf{j}\in\mathbb{B}^{p_1}} \mathbf{w}^1_{\mathbf{j}}(\eta)F^1_{\mathbf{j}}\eta(t), \tag{15}$$

where $F^1_{\mathbf{j}} \in \mathbb{R}^{n_2\times n_1}$, $\mathbf{j} \in \mathbb{B}^{p_1}$, are gains to be found. Substituting (15) in (14) produces the closed-loop system

$$\dot{\eta}(t) = \sum_{\mathbf{i}\in\mathbb{B}^{p_1}}\sum_{\mathbf{j}\in\mathbb{B}^{p_1}} \mathbf{w}^1_{\mathbf{i}}(\eta)\mathbf{w}^1_{\mathbf{j}}(\eta)\left(A^1_{\mathbf{i}} + B^1_{\mathbf{i}}F^1_{\mathbf{j}}\right)\eta(t). \tag{16}$$

We can now state our first result:

Theorem 1: The origin $\eta = 0$ of the closed-loop system (16) is asymptotically stable if there exists matrices $X_1 \in \mathbb{R}^{n_1\times n_1}$ and $M^1_{\mathbf{j}} \in \mathbb{R}^{n_2\times n_1}$, $\mathbf{j} \in \mathbb{B}^{p_1}$, such that the LMIs

$$X_1 > 0, \ \mathrm{He}\left(A^1_{\mathbf{i}}X_1 + B^1_{\mathbf{i}}M^1_{\mathbf{j}}\right) < 0, \tag{17}$$

hold for every $\mathbf{i}$, $\mathbf{j} \in \mathbb{B}^{p_1}$. In that case, the controller gains are $F^1_{\mathbf{j}} = M^1_{\mathbf{j}}X_1^{-1}$, $\mathbf{j} \in \mathbb{B}^{p_1}$, and any trajectory within the outermost Lyapunov level $V_1(\eta) = \eta^T P_1 \eta \le c$, $P_1 = X_1^{-1}$, $c > 0$, within the compact of interest Ω_1 will converge asymptotically to $\eta = 0$.

Proof 1: If LMI conditions (17) hold, it is clear that $V_1(\eta) = \eta^T P_1\eta$ is a Lyapunov function candidate since $P_1 = X_1^{-1} > 0$; taking into account the closed-loop system (16), its time derivative can be written as

$$\begin{aligned}\dot{V}_1 =& \mathrm{He}\left(\eta^T P_1\dot{\eta}\right) = \\ =& \sum_{\mathbf{i}\in\mathbb{B}^{p_1}}\sum_{\mathbf{j}\in\mathbb{B}^{p_1}} \mathbf{w}^1_{\mathbf{i}}(\eta)\mathbf{w}^1_{\mathbf{j}}(\eta)\eta^T\mathrm{He}\left(P_1A^1_{\mathbf{i}} + P_1B^1_{\mathbf{i}}F^1_{\mathbf{j}}\right)\eta,\end{aligned}$$

which is negative definite for $\eta \in \Omega_1$ if

$$\mathrm{He}\left(P_1A^1_{\mathbf{i}} + P_1B^1_{\mathbf{i}}F^1_{\mathbf{j}}\right) < 0$$

for every $\mathbf{i}$, $\mathbf{j} \in \mathbb{B}^{p_1}$, because $0 \le \mathbf{w}^1_{\mathbf{i}}(\eta)\mathbf{w}^1_{\mathbf{j}}(\eta) \le 1$ in Ω_1. But the second set of LMIs (17) is equivalent to the previous inequalities as can be seen pre- and post-multiplying them by $X_1 = P_1^{-1}$ and renaming $F^1_{\mathbf{j}}X_1$ as $M^1_{\mathbf{j}}$. Thus, $V_1(\eta)$ is a valid Lyapunov function establishing asymptotic stability of the origin $\eta = 0$ of the closed-loop system (16). Since $V_1(\eta)$ holds the Lyapunov function requirements for $\eta \in \Omega_1$, it follows that any trajectory within the outermost Lyapunov level $V_1(\eta) = \eta^T P_1\eta \le c$, $P_1 = X_1^{-1}$, $c > 0$, within the compact of interest Ω_1 will converge asymptotically to $\eta = 0$, thus concluding the proof. □.

We now go back to our original system (12)-(13). As customary within the backstepping framework, we add and subtract $g_1(\eta)F(\eta)\eta(t)$ on the righthand side of (12) as to obtain the equivalent system:

$$\dot{\eta}(t) = f_1(\eta) + g_1(\eta)F^1(\eta)\eta(t) + g_1(\eta)\zeta(t), \tag{18}$$

$$\dot{\zeta}(t) = \bar{f}_2(\eta,\xi) + g_2(\eta,\xi)u(t), \tag{19}$$

where $\zeta = \xi - F^1(\eta)\eta$ and

$$\bar{f}_2(\eta,\xi) = f_2(\eta,\xi) - \frac{\partial F^1(\eta)\eta}{\partial\eta}\left(f_1(\eta) + g_1(\eta)F(\eta)\eta + g_1(\eta)\zeta\right).$$

Clearly, if $F^1(\eta)$ is calculated as the convex sum in (15) whose gains hold the conditions of Theorem 1, the origin $\eta = 0$ of the first subsystem above is asymptotically stable when $\zeta = 0$.

Again, by means of convex embedding, (18)-(19) can be rewritten as

$$\begin{aligned}\begin{bmatrix}\dot{\eta}(t)\\ \dot{\zeta}(t)\end{bmatrix} &= \begin{bmatrix}A^2(\eta) & B^1(\eta)\\ A^3(\eta,\zeta) & A^4(\eta,\zeta)\end{bmatrix}\begin{bmatrix}\eta(t)\\ \zeta(t)\end{bmatrix} + \begin{bmatrix}0\\ B^2(\eta,\zeta)\end{bmatrix}u(t) \\ &= \sum_{\mathbf{i}\in\mathbb{B}^{p_2}} \mathbf{w}^2_{\mathbf{i}}(\eta,\zeta)\left(\begin{bmatrix}A^2_{\mathbf{i}} & B^1_{\mathbf{i}}\\ A^3_{\mathbf{i}} & A^4_{\mathbf{i}}\end{bmatrix}\begin{bmatrix}\eta(t)\\ \zeta(t)\end{bmatrix} + \begin{bmatrix}0\\ B^2_{\mathbf{i}}\end{bmatrix}u(t)\right),\end{aligned} \tag{20}$$

where $A^2(\eta)\eta = A^1(\eta)\eta + B^1(\eta)F^1(\eta)\eta$, $A^3(\eta,\zeta)\eta + A^4(\eta,\zeta)\zeta = \bar{f}_2(\eta,\xi)$, $B^2(\eta,\zeta) = g_2(\eta,\xi)$, p_2 is the number of different non-constant terms in $A^2(\eta)$, $B^1(\eta)$, $A^3(\eta,\zeta)$, $A^4(\eta,\zeta)$, and $B^2(\eta,\zeta)$ which are bounded if (η,ζ) belongs to a compact set $\Omega_2 \subset \mathbb{R}^{n_1+n_2}$ containing $(\eta,\zeta) = (0,0)$; $\mathbf{w}^2_{\mathbf{i}}(\eta,\zeta)$ are constructed from these terms; $A^2_{\mathbf{i}} = A^2(\eta)|_{\mathbf{w}^2_{\mathbf{i}}=1}$, $A^3_{\mathbf{i}} = A^3(\eta,\zeta)|_{\mathbf{w}^2_{\mathbf{i}}=1}$, $A^4_{\mathbf{i}} = A^4(\eta,\zeta)|_{\mathbf{w}^2_{\mathbf{i}}=1}$, and $B^2_{\mathbf{i}} = B^2(\eta,\zeta)|_{\mathbf{w}^2_{\mathbf{i}}=1}$, $\mathbf{i} \in \mathbb{B}^{p_2}$.

Naturally, applying a PDC control law

$$u(t) = F^2(\eta,\zeta)\begin{bmatrix}\eta(t)\\ \zeta(t)\end{bmatrix} \equiv \sum_{\mathbf{j}\in\mathbb{B}^{p_2}} \mathbf{w}_{\mathbf{j}}^2(\eta,\zeta)F_{\mathbf{j}}^2\begin{bmatrix}\eta(t)\\ \zeta(t)\end{bmatrix}, \tag{21}$$

where $F_{\mathbf{j}}^2 \in \mathbb{R}^{m\times(n_1+n_2)}$, $\mathbf{j} \in \mathbb{B}^{p_2}$, are the new gains to be found, to (21) in (20), produces the closed-loop system (omitting arguments when convenient)

$$\begin{bmatrix}\dot\eta\\ \dot\zeta\end{bmatrix} = \sum_{\mathbf{i}\in\mathbb{B}^{p_2}}\sum_{\mathbf{j}\in\mathbb{B}^{p_2}} \mathbf{w}_{\mathbf{i}}^2(\eta,\zeta)\mathbf{w}_{\mathbf{j}}^2(\eta,\zeta)\left(\begin{bmatrix}A_{\mathbf{i}}^2 & B_{\mathbf{i}}^1\\ A_{\mathbf{i}}^3 & A_{\mathbf{i}}^4\end{bmatrix} + \begin{bmatrix}0\\ B_{\mathbf{i}}^2\end{bmatrix}F_{\mathbf{j}}^2\right)\begin{bmatrix}\eta\\ \zeta\end{bmatrix}. \tag{22}$$

Thus, the following theorem synthesizing u can be stated:

Theorem 2: The origin $(\eta,\zeta) = (0,0)$ of the closed-loop system (22) is asymptotically stable if there exists matrices $X_2 \in \mathbb{R}^{(n_1+n_2)\times(n_1+n_2)}$ and $M_{\mathbf{j}}^2 \in \mathbb{R}^{m\times(n_1+n_2)}$, $\mathbf{j} \in \mathbb{B}^{p_2}$, such that the LMIs

$$X_2 > 0,\ \mathrm{He}\left(\begin{bmatrix}A_{\mathbf{i}}^2 & B_{\mathbf{i}}^1\\ A_{\mathbf{i}}^3 & A_{\mathbf{i}}^4\end{bmatrix}X_2 + \begin{bmatrix}0\\ B_{\mathbf{i}}^2\end{bmatrix}M_{\mathbf{j}}^2\right) < 0, \tag{23}$$

hold for every $\mathbf{i}$, $\mathbf{j} \in \mathbb{B}^{p_2}$. In that case, the controller gains are $F_{\mathbf{j}}^2 = M_{\mathbf{j}}^2 X_2^{-1}$, $\mathbf{j} \in \mathbb{B}^{p_2}$, and any trajectory within the outermost Lyapunov level $V_2(\eta,\zeta) = [\eta^T\ \zeta^T]P_2[\eta^T\ \zeta^T]^T \leq c$, $P_2 = X_2^{-1}$, $c > 0$, within the compact of interest Ω_2 will converge asymptotically to $(\eta,\zeta) = (0,0)$.

Proof 2: It follows the same lines as that of Theorem 1.

Remark 1: Theorems 1 and 2 can be applied iteratively to systems that have more than two decompositions (η and ξ) by repeating the procedure described therein. Note that it is possible for certain systems to be decomposed in several ways, depending on their structure.

Remark 2: Since theorems 1 and 2 are LMI-based, they can incorporate performance specifications in a straightforward manner, e.g., decay rate, input/output constraints, $\mathscr{H}_\infty$, and other. The interested reader is referred to [9].

IV. Examples

Example 1 (Single-input system): Consider stabilizing the following system in the normal form:

$$\dot\eta = \eta^2 - \eta^3 + \xi, \tag{24}$$
$$\dot\xi = u. \tag{25}$$

Following the methodology described in the previous section, (24) can be written as

$$\dot\eta = (\eta(1-\eta))\eta + \xi,$$

where $A^1(\eta) = \eta(1-\eta)$ and $B^1(\eta) = 1$; the non-constant terms in these expressions are $z_1(\eta) = \eta$ and $z_2(\eta) = 1-\eta$.

Assume we are interested in stabilizing form initial conditions in $\eta \in [-0.5, 0.5]$; then, the non-constant terms are bounded as $z_1 \in [-0.5, 0.5]$ and $z_2 \in [0.5, 1.5]$. Consequently, the convex functions used to rewrite the system as a convex sum are

$$w_0^1(\eta) = 0.5 - z_1(\eta), \qquad w_1^1(\eta) = 1 - w_0^1(\eta),$$
$$w_0^2(\eta) = 1.5 - z_2(\eta), \qquad w_1^2(\eta) = 1 - w_0^2(\eta).$$

Therefore the vertex matrices $A_{\mathbf{i}}^1$, $\mathbf{i} \in \mathbb{B}^2$, are

$$A_{00}^1 = -0.25,\ A_{01}^1 = -0.75,\ A_{10}^1 = 0.25,\ A_{11}^1 = 0.75,$$

and $B_{\mathbf{i}}^1 = 1$ as it was constant from the beginning.

Using the $V_1 = \eta^T P_1 \eta$ as the Lyapunov function candidate with $P_1 > 0$, the LMI conditions in Theorem 1 are found feasible by using the LMI Toolbox in MATLAB, resulting in the Lyapunov matrix $P_1 = 2.08 \times 10^{-9}$ and the gains $F_{00}^1 = -0.25$, $F_{01}^1 = 0.25$, $F_{10}^1 = -0.75$, and $F_{11}^1 = -1.25$. Thus, the PDC control law (15) can be constructed and used to write the following system:

$$\dot\eta = -\frac{1}{2}\eta + \zeta, \tag{26}$$
$$\dot\zeta = \frac{3}{2}\eta^3 - \eta^2 - \frac{1}{4}\eta + (-3\eta^2 + 2\eta - \frac{1}{2})\zeta + u. \tag{27}$$

which is equivalent to (24)-(25), where $\zeta = \xi - F^1(\eta)\eta$.

Rewriting (26)-(27) as:

$$\begin{bmatrix}\dot\eta\\ \dot\zeta\end{bmatrix} = \begin{bmatrix}-\frac{1}{2} & 1\\ \frac{3}{2}\eta^2 - \eta - \frac{1}{4} & 2\eta(1-\frac{3}{2}\eta) - \frac{1}{2}\end{bmatrix}\begin{bmatrix}\eta\\ \zeta\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u, \tag{28}$$

we can now perform convex embedding as before by considering the region of interest $\eta \in [-0.5, 0.5]$. Thus, $z_1 = \eta \in [-0.5, 0.5]$ and $z_2 = (1 - \frac{3}{2}\eta) \in [0.25, 1.75]$, which, together with

$$w_0^1 = 0.5 - z_1(x), \qquad w_1^1 = 1 - w_0^1,$$
$$w_0^2 = \frac{1.75 - z_2(x)}{1.5}, \qquad w_1^2 = 1 - w_0^2,$$

allow obtaining a convex model of the form (20) with $(A_{\mathbf{i}}^2, A_{\mathbf{i}}^3, A_{\mathbf{i}}^4, B_{\mathbf{i}}^1, B_{\mathbf{i}}^2)$, $\mathbf{i} = (0,0), (0,1), (1,0), (1,1)$ as

$$(-0.5, -0.125, 0.25, 1, 1), (-0.5, -0.625, -1.25, 1, 1),$$
$$(-0.5, -0.375, 0.75, 1, 1), (-0.5, -1.125, 2.25, 1, 1).$$

Given the Lyapunov function candidate $V_2(\eta,\zeta) = [\eta^T\ \zeta^T]P_2[\eta^T\ \zeta^T]^T$, with $P_2 = P_2^T > 0$, and a PDC control law of the form (21), gains $F_{\mathbf{j}}^2$ stabilizing (28) can be found via the LMI Toolbox in MATLAB with Theorem 2; it yields $P_2 = \mathrm{diag}(0.4172 \times 10^{-8}, 0.4172 \times 10^{-8})$ and gains

$$F_{00}^2 = \begin{bmatrix}-0.875 & -0.75\end{bmatrix}, \qquad F_{01}^2 = \begin{bmatrix}-1.625 & 0.75\end{bmatrix},$$
$$F_{10}^2 = \begin{bmatrix}-0.625 & -1.25\end{bmatrix}, \qquad F_{11}^2 = \begin{bmatrix}0.125 & -2.75\end{bmatrix},$$

which allow constructing the final control law of the form (25) as

$$u = -3\eta^5 + 5\eta^4 - \eta^3 - \eta^2 - \frac{5}{4}\eta - \xi - 2\eta\xi + 3\eta^2\xi, \tag{29}$$

where $\zeta = \xi - \eta^3 + \eta^2 + \frac{1}{2}\eta$ has been used to write the expression in terms of the original state variables.

Fig. 1 shows the time evolution of the states with initial conditions $\eta(0) = 0.3$ and $\xi(0) = -1$ (top) when the system is subject to control law (29) (top, solid lines); the corresponding control signals are shown in Fig. 1 (bottom, solid lines). When ordinary backstepping is employed the signals in dashed lines are obtained; although settling times are similar, the control effort is lower in our approach than in the ordinary one.

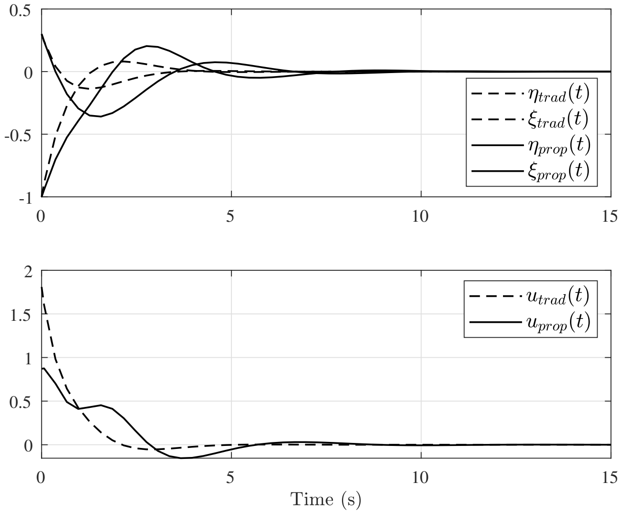

Fig. 1. Time evolution of states and control signal in Example 1.

Example 2 (SCARA robot manipulator): Consider the task of stabilizing the SCARA robot manipulator in Fig. 2, which consists in two links with rotational joints; its dynamical model is given by [22]:

$$\dot{\eta}(t) = \xi(t) \tag{30}$$

$$\dot{\xi}(t) = \begin{bmatrix} c_2c_4c_5\xi_1^2 + c_3c_4c_5\xi_1\xi_2 + c_3c_4c_5\xi_2^2 \\ -c_1c_4c_5\xi_1^2 - c_2c_4c_5\xi_1\xi_2 - c_2c_4c_5\xi_2^2 \end{bmatrix} + \begin{bmatrix} c_3c_4 & -c_2c_4 \\ -c_2c_4 & c_1c_4 \end{bmatrix} u(t), \tag{31}$$

where $\eta = [\eta_1 \;\; \eta_2]^T \equiv [\theta_1 \;\; \theta_2]^T$ and $\xi = [\xi_1 \;\; \xi_2]^T \equiv [\dot{\theta}_1 \;\; \dot{\theta}_2]^T$ are positions and angular velocities, respectively; $u = [u_1 \;\; u_2]^T$, being u_1 and u_2 torques applied to the first and second link, respectively; $c_1 = J_{m_1} + J_{HD_1} + k^2(m_1l^2 + 2m_2l^2(1+\cos\eta_2))$, $c_2 = k^2m_2l^2(1+\cos\eta_2)$, $c_3 = J_{m_2} + J_{HD_2} + k^2m_2l^2$, $c_4 = 1/(c_1c_3 - c_2^2)$, $c_5 = km_2l^2\sin\eta_2$. The plant parameters are specified in the Table I. Note that ordinary backstepping cannot be applied to MIMO plants.

Applying the first step of the methodology in the previous section, subsystem (30) needs to be artificially stabilized by a virtual input ξ, which needs to be calculated by convex modelling and LMIs. Since this subsystem is linear with constant matrices $A^1(\eta) = 0$ and $B^1(\eta) = I$, Theorem 1 can

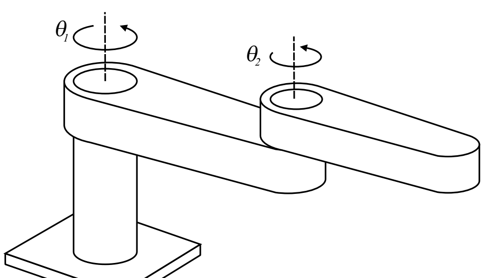

Fig. 2. SCARA robot manipulator.

TABLE I
PARAMETERS OF THE SCARA MANIPULATOR.

Parameter	Symbol	Value
Mass of link 1	m_1	9.55 kg
Mass of link 2	m_2	12.12 kg
Lenght of link 1 and 2	l	0.33 m
Inertial moment of link 1	J_{m1}	0.88×10^{-4} kg·m^2
Inertial moment of link 2	J_{m2}	1.67×10^{-4} kg·m^2
Inertial moment of motor 1	J_{HD1}	5.45×10^{-4} kg·m^2
Inertial moment of motor 2	J_{HD2}	2.01×10^{-4} kg·m^2
Reduction rate of the gears	k	0.01

be directly applied without any need of convex modelling; it yields a feasible solution from which the following is obtained:

$$P_1 = \begin{bmatrix} 0.1890 & 0 \\ 0 & 0.1890 \end{bmatrix} \times 10^{-8}, \; F^1 = \begin{bmatrix} -0.5 & 0 \\ 0 & -0.5 \end{bmatrix}.$$

At this point, the origin $\eta = 0$ of subsystem (30) is asymptotically stable. Applying the change of variable $\zeta = \xi - F^1\eta$ to (30)-(31) allows writing the model in the form of the leftmost expression in (20) with matrices (omitting arguments):

$$A^2 = -0.5I, \;\; B^1 = I, \;\; B^2 = \begin{bmatrix} c_3c_4 & -c_2c_4 \\ -c_2c_4 & c_1c_4 \end{bmatrix},$$

$$A^3 = -\frac{1}{2}\begin{bmatrix} c_2c_4c_5(\zeta_1 - 0.5\eta_1) - \frac{1}{4} & \begin{pmatrix} c_3c_4c_5(\zeta_1 - 0.5\eta_1) \\ +c_3c_4c_5(\zeta_2 - 0.5\eta_2) \end{pmatrix} \\ -c_1c_4c_5(\zeta_1 - 0.5\eta_1) & \begin{pmatrix} c_2c_4c_5(\zeta_1 - 0.5\eta_1) \\ -c_2c_4c_5(\zeta_2 - 0.5\eta_2) - \frac{1}{4} \end{pmatrix} \end{bmatrix},$$

$$A^4 = \begin{bmatrix} c_2c_4c_5(\zeta_1 - 0.5\eta_1) + \frac{1}{2} & \begin{pmatrix} c_3c_4c_5(\zeta_1 - 0.5\eta_1) \\ +c_3c_4c_5(\zeta_2 - 0.5\eta_2) \end{pmatrix} \\ -c_1c_4c_5(\zeta_1 - 0.5\eta_1) & \begin{pmatrix} c_2c_4c_5(\zeta_1 - 0.5\eta_1) \\ -c_2c_4c_5(\zeta_2 - 0.5\eta_2) + \frac{1}{2} \end{pmatrix} \end{bmatrix}.$$

Convex embedding of the non-constant terms in the matrices above is now performed; they are listed in Table II along with their bounds. These bounds are induced by the region on which we want to perform stabilization, which is $\eta_1, \eta_2 \in [-\pi/3, \pi/3]$ and $\zeta_1, \zeta_2 \in [-0.2, 0.2]$. For illustration purposes only two tuples $(A^2_{\mathbf{i}}, A^3_{\mathbf{i}}, A^4_{\mathbf{i}}, B^1_{\mathbf{i}}, B^2_{\mathbf{i}})$ out of 2^6 are displayed, namely, $\mathbf{i} = (0,0,0,0,0,0)$ and $\mathbf{i} = (1,0,1,0,1,0)$:

$$\left(-0.5I, \begin{bmatrix} -1.7048 & -7.3482 \\ 8.3255 & 2.6597 \end{bmatrix}, \begin{bmatrix} 3.4097 & 14.6965 \\ -16.6510 & -5.3194 \end{bmatrix}, I, \begin{bmatrix} 0.8884 & -0.3518 \\ -0.3518 & 2.0132 \end{bmatrix} \times 10^3\right), \left(-0.5I, \begin{bmatrix} -1.7048 & -7.3482 \\ 8.3255 & 2.6597 \end{bmatrix}, \begin{bmatrix} 3.4097 & 14.6965 \\ -16.6510 & -5.3194 \end{bmatrix}, I, \begin{bmatrix} 0.8884 & -0.3518 \\ -0.3518 & 2.0132 \end{bmatrix} \times 10^3\right).$$

TABLE II
DEFINITIONS OF NON-CONSTANT TERMS

Term	Definition	Lower bounds z_i^0	Upper bounds z_i^1
z_1	c_1	1.1329×10^{-3}	1.2649×10^{-3}
z_2	c_2	0.1979×10^{-3}	0.2639×10^{-3}
z_3	c_4	1.7769×10^{6}	1.8965×10^{6}
z_4	c_5	-0.0114	0.0114
z_5	$\zeta_1 - 0.5\eta_1$	-0.7236	0.7236
z_6	$\zeta_2 - 0.5\eta_2$	-0.7236	0.7236

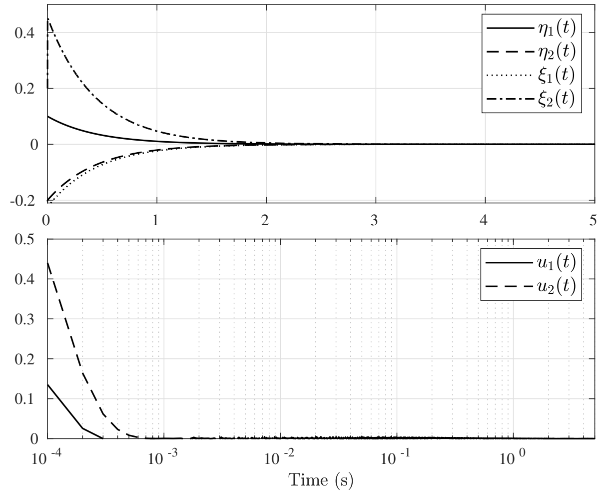

Fig. 3. Time evolution of states and control signal in Example 2.

Solving LMIs in Theorem 2 with a decay rate of $\alpha = 0.05$ produces a feasible solution from which the gains $F^2_{\mathbf{j}}$, $\mathbf{j} \in \mathbb{B}^6$, are calculated; again, for brevity, only 2 of the 2^6 gains are displayed:

$$F^2_{000000} = \begin{bmatrix} -35.6983 & -21.1873 & -12.1869 & -7.9054 \\ -5.3467 & -11.1528 & -1.1005 & -5.3460 \end{bmatrix},$$
$$F^2_{111111} = \begin{bmatrix} -29.8117 & -3.1402 & -11.4987 & 0.9899 \\ -15.1554 & -11.5306 & -4.9439 & -4.6787 \end{bmatrix}.$$

Constructing convex functions

$$\mathbf{w}^2_{\mathbf{j}} = w^1_{j_1}(z_1)w^2_{j_2}(z_2)w^3_{j_3}(z_3)w^4_{j_4}(z_4)w^5_{j_5}(z_5)w^6_{j_6}(z_6),$$
$$w^1_0(z_1) = \frac{z^1_1 - z_1}{z^1_1 - z^0_1} = \frac{1.2649 \times 10^{-3} - c_1}{1.2649 \times 10^{-3} - 1.1329 \times 10^{-3}}$$
$$w^2_0(z_2) = \frac{z^1_2 - z_2}{z^1_2 - z^0_2} = \frac{0.2639 \times 10^{-3} - c_2}{0.2639 \times 10^{-3} - 0.1979 \times 10^{-3}}$$
$$w^3_0(z_3) = \frac{z^1_3 - z_3}{z^1_3 - z^0_3} = \frac{1.8965 \times 10^{6} - c_4}{1.8965 \times 10^{6} - 1.7769 \times 10^{6}}$$
$$w^4_0(z_4) = \frac{z^1_4 - z_4}{z^1_4 - z^0_4} = \frac{0.0114 - c_5}{0.0114 - (-0.0114)}$$
$$w^5_0(z_5) = \frac{z^1_5 - z_5}{z^1_5 - z^0_5} = \frac{0.7236 - (\zeta_1 - 0.5\eta_1)}{0.7236 - (-0.7236)}$$
$$w^6_0(z_6) = \frac{z^1_6 - z_6}{z^1_6 - z^0_6} = \frac{0.7236 - (\zeta_2 - 0.5\eta_2)}{0.7236 - (-0.7236)},$$

$w^i_1(z_i) = 1 - w^i_0(z_i)$, $i \in \{1, 2, \ldots, 6\}$, the control law (21) is completed; once applied the system states and control signals shown in Fig. 3 are obtained with initial conditions $\eta_1(0) = 0.1$, $\eta_2(0) = -0.2$, $\xi_1(0) = -0.2$ and $\xi_2(0) = 0.2$.

V. Conclusions

A systematic method for backstepping control design via LMIs has been presented. It has been shown that finding a control law as a convex sum of gains within a compact set of interest at each step of the backstepping methodology, can be done via LMIs, guaranteeing that any trajectory within the outermost Lyapunov level within such compact converges asymptotically to the origin. It has been shown that the proposed method can straightforwardly deal with MIMO systems as it makes no difference for the required matrix decoupling, providing an advantage over the classical methodology of backstepping.

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