Nonlinear Modeling and Least Squares Optimization on Consolidation Property of Composite Soil Samples

Hang Yin, Zhengmao Ye, Huey Lawson, Albertha Lawson
College of Science and Engineering, Southern University, Baton Rouge, LA70813, USA
hang_yin@subr.edu, zhengmao_ye@subr.edu, huey_lawson@subr.edu, albertha_lawson@subr.edu

Abstract — **Nonlinear least square estimation is proposed to determine the physical index and self-weight consolidation properties of diverse composite samples being homogenized out of tube samples, which are taken from various borings in the State of Louisiana. Due to the high sensitivity of soil, a low pressure loading system is applied to perform self-weight consolidation tests, with stresses ranging from 10, 50, 100, 200 to 600 psf and one unloading segment at 200 psf. Typical data analysis using the Casagrande (Log time) method or Taylor (Root time) method could be implemented to determine the primary parameter of the coefficient of consolidation (Cv), which would require long-lasting multiple procedures to locate a few consolidation points on testing data curves manually. A simple hyperbolic tangent function model instead is proposed in this study. Parametric optimization is conducted via nonlinear least square estimation, where data points from 0 to 100% deformation will all be used to optimize the result and compute the coefficient of consolidation (Cv), in order to better predict the actual time rate of the settlement. It indicates that those composite samples are in fact the well degraded clays with the high in-situ water contents, high plasticity, low consolidation coefficients and low hydraulic conductivities. For testing results, coefficients of consolidation obtained from the Taylor method is slightly higher than the Casagrande method. Least square curve fitting has the potential to provide more accurate results than the two existing methods.**

Keywords*: *Nonlinear Modeling, Least Square Estimation, Optimization, Hyperbolic Function, Physical Index, Coefficient of Consolidation

I. INTRODUCTION

Nonlinear numerical approaches can be broadly applied to various engineering fields. For example, nonlinear kernel based component analysis has been employed in the complex flocculation mechanism analysis. The impact of salinity on mechanical behaviors of flocs has been examined [1]. In this study, consolidation properties of composite soil samples will be conducted.

The land mass along the coastal Louisiana is undergoing rapid deterioration stem from both interior marsh and wetland losses and shoreline erosion. These land losses have been widely recognized on different sites along the coastal zone, such as the northern banks of Terrebonne Bay, the northeast of Lost Lake, the west of Lost Lake, the perimeter of Sweetbay Lake, the Black Lake and Brown Lake, and the north of Terrebonne Bay, etc. Among these locations, emergent marshes at north of Terrebonne Bay have been eroding as fast as or even faster than almost any other marshes along coastal Louisiana with high interior landloss rates calculated to be 2% per year and moderate shoreline erosion rates calculated to be between 3 and 8 ft per year. Diverse reasons may lead to the land mass loss along the coastal Louisiana, which include but not limited to: (1) a lack of sediment input and a limited supply of freshwater coupled with past dredging of oil and gas canals; (2) increasing open-water fetch conditions and marine traffic; (3) direct removal of emergent marsh by severe hurricanes (e.g. Katrina and Ida); (4) ponding and periodic entrapment of higher salinity waters during storm events; and (5) block of freshwater and sediment flow avenues as well as flow circulating short-circuited due to the construction of the ship channel, etc. The loss in the marsh, and wetland as well as the shoreline erosion has given rise to significant environmental problems, such as the flooding problems, increased tidal prism as well as interior marsh loss acceleration. In order to reduce the salt water intrusion and storm surge, marsh restoration and wetland creation projects are designed along the coastal Louisiana to stabilize the land mass and slow down the progress of high saline waters, by borrowing material from readily available sediments and nutrients in a previously degraded marsh area and pumped via a hydraulic dredge into the marsh creation sites. Thus, it is important to have a thorough knowledge of the evolution of the physical and hydraulic properties of such slurried depositions. Several laboratory tests have been performed on these composite slurry soil samples, including the water content, specific gravity, Atterberg limits, and self-weight consolidation, to determine the physical indices and consolidation properties of diverse composite soil samples. On the other hand, those laboratory works will provide practical materials to solve real world problems for the higher education as well [2-9].

Both linear and nonlinear least square estimation methods could be applied to parametric optimization in engineering problems. For instance, the linear least square estimation has been used to automotive engine transient fuel control problem in order to determine two typical parameters in wall wetting models. In another case study, nonlinear least square regression analysis has been applied to measurement systems via Laser doppler instrumentation and Raman spectroscopy, respectively,

978-1-6654-0029-9/21/$31.00 ©2021 IEEE

so as to eliminate unavoidable mixing artifacts [10-11]. In this work, hyperbolic tangent function is proposed to model the curve in the Casagrande (Log time) method, then the nonlinear least square approach is implemented to optimize a set of parameters, including the elapse time (t_{50}) taken for soil to reach 50% primary consolidation and its corresponding vertical deformation (d_{50}) relevant 50% primary consolidation. The preliminary comparison between the optimal result and those from two graphical methods have been made. It shows the feasibility of this novel approach.

II. METHODS

All the experiments are scheduled under the instruction of American Society for Testing and Materials (ASTM) standards. The associated ASTM numbers used in these experiments are listed in Table 1on physical index and consolidation properties.

TABLE 1. ASSOCIATED ASTM NUMBERS

Physical Index and Consolidation Properties	ASTM Number
Water Content, ω	D-2216
Organic Content	D-2974
Specific Gravity, *S*	D-854
Atterberg Limits (LL, PL, and PI)	D-4318

The tube samples are taken from different borings and then homogenized in the lab. Water content (ω) and specific gravity (S) are firstly measured on the homogenized sample. Prior to the Atterberg limits and consolidation tests, the mixed slurry samples are carefully examined to manually remove the gravel-sized debris like the broken shells, etc. The homogenized composite samples were diluted evenly to (1.0-1.5) times the liquid limit (ω_L) of samples. It is to ensure that the sample has been reconstituted to a slurry consistency allowing the free flow under gravity, which eases sample preparation. The water content of the prepared initial sample slurry for three composite samples is also measured and recorded for future calculation. Owing to the high sensitivity of the soil, the low pressure one-dimensional incremental loading method is used for the consolidation, with stresses ranging from 10, 25, 50, 100, 200 to 600 psf and one unloading segment at 200 psf. Both the Casagrande (Log time) and Taylor (Root time) methods have been employed for data analysis in order to determine the coefficient of consolidation (Cv).

III. BASIC PHYSICAL INDICES

Those basic physical indices for composite soil samples are summarized in Table 2, including the specific gravity, both the water and organic contents, as well as the Atterberg limits. It indicates the medium to high plasticity according to the plasticity chart for all the composite samples (Fig. 1).

TABLE 2. SUMMARY OF ASSOCIATED INDEX PROPERTIES

Sample ID	Specific gravity S	Water content ω(%)	Organic content (%)	Liquid limit ω_L(%)	Plastic limit ω_P (%)	Plastic index I_P (%)
Site 1	2.66	119.04	9.57	115.7	33.9	81.8
2-CS1	2.64	59.33	2.99	41.9	24.5	17.4
2-CS2	2.62	49.10	2.30	34.2	19.5	14.7
2-CS3	2.60	67.31	4.16	47.1	24.2	22.9
3-TOP	2.42	236.81	14.11	185.8	64.2	121.6
3-MID	2.35	210.87	13.55	190.0	65.7	124.3
3-BTM	2.47	210.95	12.21	185.4	66.5	118.9

In Fig. 1, the high plasticity index comes with the relative high organic content. The result shows that those composites are well degraded.

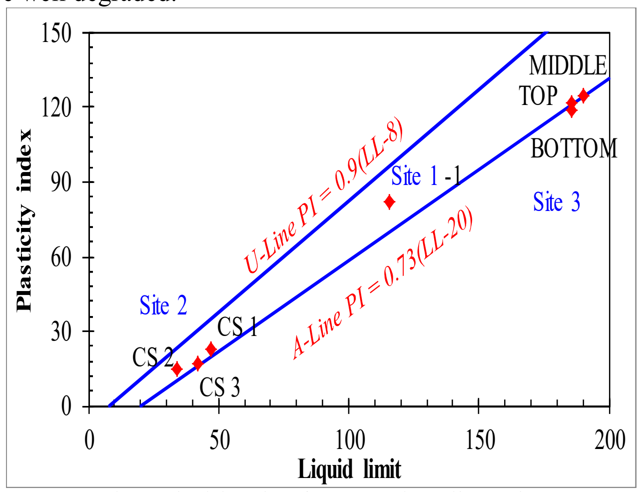

Fig. 1. Plasticity Chart for Composite Soil Samples

IV. SELF-WEIGHT CONSOLIDATION LAB TEST

The self-weight consolidation lab tests have also been conducted accordingly. Fig. 2 and Fig. 3 actually illustrate the representative self-weight consolidation results evaluated by two methods: (a) the Casagrande (Log time), and (b) the Taylor (Root time). The data were recorded under the loading segment with a stress of 25 psf applied onto the specimen homogenized from the sample in the Site 1.

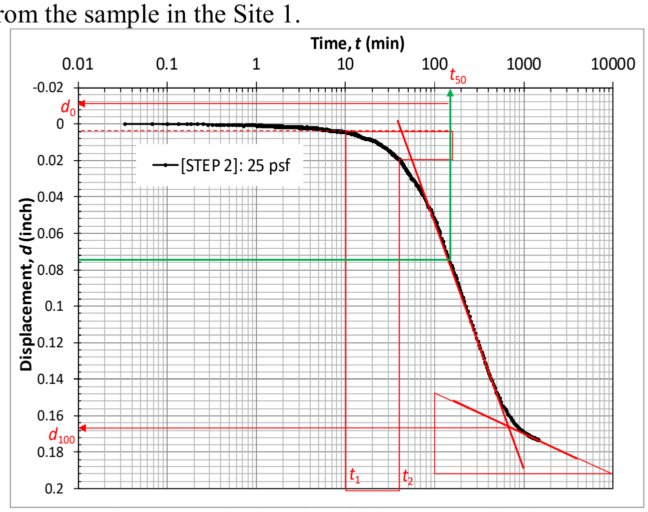

Fig. 2. Determination of Coefficient of Consolidation
(a) Casagrande (Log Time) Method

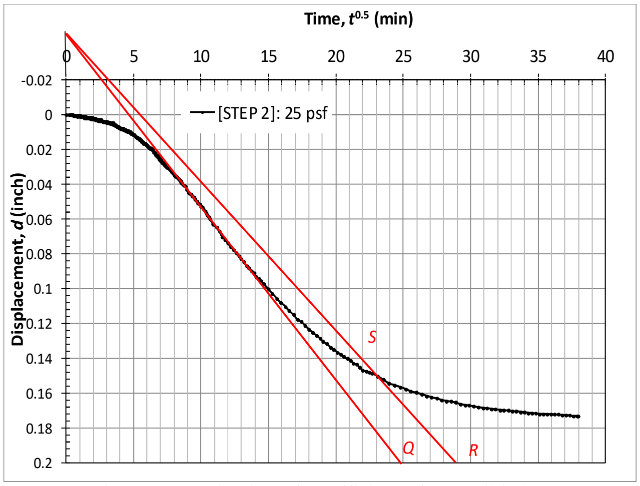

Fig. 3. Determination of Coefficient of Consolidation (b) Taylor (Root Time) Method

The composites collected from different sites are tested. For all composite samples in different locations evaluated using 2 methods of the Log time and Root time, the trends of the coefficient of consolidation (C_v) along with consolidation pressure are graphically depicted in the Fig. 4 and Fig. 5, respectively. It is clearly seen that for all the three composites, there is a general trend of slightly increment in a rate of primary consolidation under small consolidation pressure (e.g., from 25 to 50 psf); followed by a slight reduction in the consolidation rate once the effective stress goes beyond 50 psf. By comparison among the 3 sites, it also indicates that the C_v for 3 samples in the Site 1 ranges from 0.9 to 1.9 ft^2/year, which is lower than that of the Site 2, 1.5 to 4.5 ft^2/year, for a range of consolidation pressure of 25 to 600 psf. Under each consolidation pressure from these tests, the coefficient of consolidation have been shown.

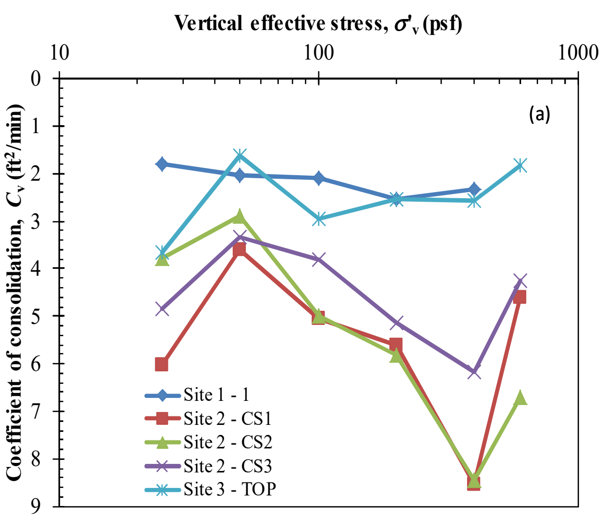

Fig. 4. Coefficient of Consolidation and Vertical Effective Stress for Composite Samples: (a) (Log Time) Method

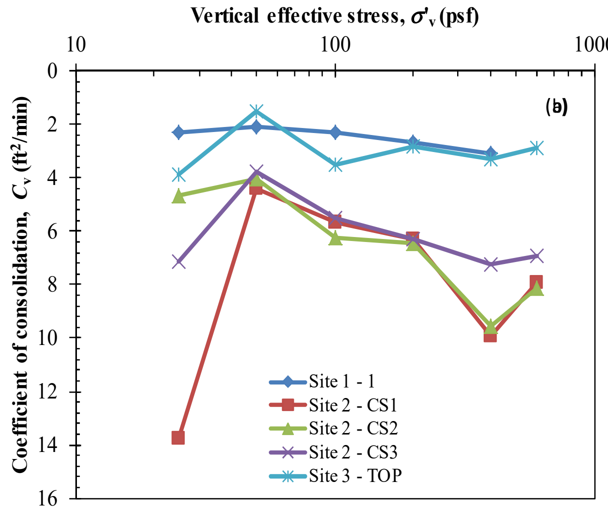

Fig. 5. Coefficient of Consolidation and Vertical Effective Stress for Composite Samples: (b) Taylor (Root Time) Method

V. MODELING ON CONSOLIDATION

The popular one-dimension consolidation model turns out to be relatively simple, with some assumptions of the constant total stress (σ), constant permeability coefficient (K) for the vertical flow, as well as the constant volume compressibility coefficient (m_v). It covers the rectilinear element of soil, so that vertical direction seepage flow takes place. It could be then formulated as the partial differential equation in (1).

$$\frac{\partial u}{\partial t} = C_V \frac{\partial^2 u}{\partial z^2} \tag{1}$$

where three variables are defined. u is the excess pore pressure, z is the depth of the element in certain layer, and t is the elapse time upon loading. Its solution is shown in (2).

$$C_V = \frac{K}{m_v \gamma_w} \tag{2}$$

where the coefficient of consolidation (C_V) is expressed in terms of three parameters: the permeability coefficient (K), the volume compressibility coefficient (m_v) and the specific weight of water (γ_w).

Actually many assumptions could not be easily met in the real world problems. That is the reason that various alternative methods to determine the coefficient of consolidation are also adopted, such as testing approaches of Casagrande (Log time) method and Taylor (Root time) method discussed in the last session. Both the Casagrande (Log time) method and Taylor (Root time) method are graphical techniques which require multiple steps for estimation. In general, the Casagrande (Log time) method is more widely used than the Taylor (root time) method, which is to locate the elapse time (t_{50}) it takes for the soil to reach 50% primary consolidation as well as the corresponding vertical deformation (d_{50}) of the 50% primary consolidation. As a result, an empirical solution is frequently applied and listed as (3) in practice.

$$C_v = \frac{T_{v(50)} H_{dr}^2}{t_{50}} \approx \frac{0.197 H_{dr}^2}{t_{50}} \tag{3}$$

where $T_{V(50)}$ is the time factor obtained from the lookup table (e.g. 0.197) and H_{dr} is maximal drainage distance that equals to half of the specimen height (double drainage).

Since the time factor $T_{V(50)}$ and maximal drainage distance H_{dr} are constants for each specimen test, the coefficient of consolidation (C_V) is determined uniquely by the elapse time (t_{50}). The deformation curve generally starts from 0% deformation on top till 100% deformation on bottom. The deformation increases in downward direction on the curve rather than upward direction commonly adopted. In fact the testing curve in the Casagrande (Log time) method resembles the pattern of the hyperbolic function. Thus a nonlinear model is proposed to describe the curve in Casagrande (Log time) method in order to determine the center point (t_{50}, d_{50}) representing the elapse time and deformation with respect to 50% primary consolidation. A novel hyperbolic consolidation model has been formulated as (4), which is shown as Fig. 6.

$$\begin{aligned} d(t) - d_{50} &= \frac{A}{2}\tanh[\alpha(\log_{10}t - \log_{10}t_{50})] \\ &= \frac{A}{2}\tanh[\alpha(\log_{10}\frac{t}{t_{50}})] \end{aligned} \tag{4}$$

where the hyperbolic tangent function is defined as (5) and A is the maximal deformation being measured in the Casagrande (Log time) method. The deformation is formulated as a nonlinear function of the arbitrary elapse time t, with three parameters to determine (t_{50}, d_{50}, α). Here α is the parameter that is related to the time rate of deformation.

$$\tanh(x) = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} \tag{5}$$

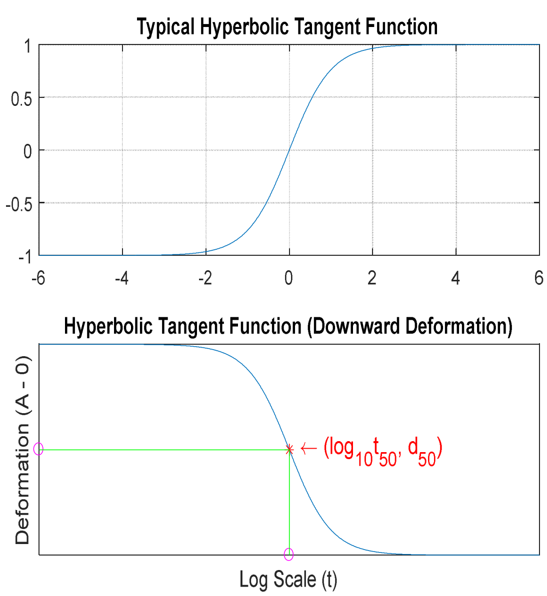

Fig. 6. Determination of Coefficient of Consolidation: Hyperbolic Consolidation Model

Especially when parameters t_{50} can be determined uniquely, the coefficient of consolidation (C_V) can also be estimated by the empirical formula in (3) easily.

VI. Optimization on Consolidation Parametrization

The proposed consolidation model can be rewritten as (6), where the 3-vector [t_{50}, d_{50}, α] covering three parameters should be optimized in order to reach the best solution t_{50} and corresponding coefficient of consolidation (C_V).

$$d(t) = \frac{A}{2}\tanh[\alpha(\log_{10}\frac{t}{t_{50}})] + d_{50} \tag{6}$$

When m sampling points (e.g. m=50 or 100) are collected to be used for the curve of Casagrande (Log time) method, m observation pairs (t_i, d_i) (i =1, 2, …, m) are available for optimization. The calculated $d(t_i)$ from the model (6) and the actual corresponding observed d_i from the Casagrande curve could have the mismatch $r(t_i)= d(t_i)–d_i$ (i =1, 2, …, m). Due to m observations, it also give rise to the form of an m-vector, representing the actual estimation error vector.

Nonlinear least square estimation is now used to minimize the estimation error. The cost function is simply defined as (7), covering all the m relevant sampling points.

$$V = \frac{1}{2}\sum_{i=1}^{m} r^2(t_i) = \frac{1}{2}\sum_{i=1}^{m}[d(t_i) - d_i]^2 \tag{7}$$

For matter of simplicity, let us redefine 3 parameters as θ_1, θ_2, θ_3, respectively, so we have (8). In fact, when parameter α is assumed to be a constant within the neighborhood of (t_{50}, d_{50}), it can further be simplified into a two-parameter estimation problem of θ_1 and θ_2, representing t_{50} and d_{50}.

$$r(t_i) = d(t_i) - d_i = d(t_{50}, d_{50}, \alpha) = d(\theta_1, \theta_2, \theta_3) \tag{8}$$

Define vector θ as [θ_1, θ_2, θ_3], the vector form of the defined cost function (7) turns out to be (9).

$$\text{Min } V(\theta_1, \theta_2, \theta_3) = \text{Min } r(\theta)^T r(\theta)/2 \to 0 \tag{9}$$

The gradient term $\nabla[V(\theta_1, \theta_2, \theta_3)]$ of the cost function is derived by the chain rule as (10).

$$\nabla[V(\theta)] = \nabla r(\theta)^T r(\theta)/2 = J(\theta)^T r(\theta) \tag{10}$$

where $J(\theta) = \nabla r(\theta)^T$ is the Jacobian of $r(\theta)$ as shown in (11).

$$J(\theta) = \begin{bmatrix} \frac{\partial r_1(\theta)}{\partial \theta_1} & \frac{\partial r_1(\theta)}{\partial \theta_2} & \frac{\partial r_1(\theta)}{\partial \theta_3} \\ \frac{\partial r_2(\theta)}{\partial \theta_1} & \frac{\partial r_2(\theta)}{\partial \theta_2} & \frac{\partial r_2(\theta)}{\partial \theta_3} \\ \vdots & \vdots & \vdots \\ \frac{\partial r_m(\theta)}{\partial \theta_1} & \frac{\partial r_m(\theta)}{\partial \theta_2} & \frac{\partial r_m(\theta)}{\partial \theta_3} \end{bmatrix} \tag{11}$$

The Hessian is computed by the chain rule again as (12).

$$\nabla^2[V(\theta)] = \nabla r(\theta)\nabla r(\theta)^T + \sum_{i=1}^{m} r_i(\theta)\nabla^2 r_i(\theta) \tag{12}$$

The Hessian of the nonlinear least squares cost function consists of one term with first order partial derivatives and other m-terms with the second-order derivatives. At each iteration, the nonlinear model could be linearized by the first-order Taylor series expansion about θ_k, while the higher-order terms are neglected. The normal equation in the matrix form is shown as (13) whose optimal solution in the least squares sense is shown as (14). If the matrix $J(\theta)^T J(\theta)$ is nonsingular, then the parametric vector $\theta=[\theta_1, \theta_2, \theta_3]$ can be determined, using the iterative formula. An initial parametric vector $\theta_0= [\theta_1, \theta_2, \theta_3]_0$

should be assigned at first, so that $[\theta_1, \theta_2, \theta_3]_1$ can be computed via the iterative formula (14), followed by $[\theta_1, \theta_2, \theta_3]_2$ and $[\theta_1, \theta_2, \theta_3]_3$, and so on. Because t_{50}^* is a key parameter to estimate, the criterion is focused on θ_1 as well. When the mismatch of first components from two vectors θ_i and θ_{i+1} is less than the specified tolerance (δ), as shown in (15), the iteration stops and in this case $\theta^*=[\theta_1, \theta_2, \theta_3]^*= [\theta_1, \theta_2, \theta_3]_{i+1}$ has been reached. On the other hand, when multiple parameters are all subject to estimation, the Euclidean norm of the difference between two vectors θ_i and θ_{i+1} can be introduced as an alternative criterion, which should be less than certain tolerance similarly, before the iteration stops.

$$[J(\theta)^T J(\theta)] (\theta_{i+1} - \theta_i)= J(\theta)^T(d_{i+1} - d_i) \tag{13}$$

$$\theta_{i+1} - \theta_i = [J(\theta)^T J(\theta)]^{-1} J(\theta)^T(d_{i+1} - d_i) \tag{14}$$

$$|[\theta_1]_i -[\theta_1]_{i+1}| <=\delta \tag{15}$$

Now the optimal first component θ_1^* also serves as the t_{50}^* to be determined. By applying the same empirical formula (3), the coefficient of consolidation (C_V) could be easily computed at the same time, based on the newly proposed nonlinear modeling and least squares optimization approach.

VII. Discussion on Numerical Results

It is necessary to compare the coefficients of consolidation determined from both the Log time method and the Root time method. In addition, the feasibility on the new hyperbolic function modeling and optimization method should also be examined. It is clearly seen from Fig. 4 and Fig. 5 that the Log method comes up with the slightly lower C_v values than the Root time method. But both are within the range of 1.8 to 2.5 ft^2/min under the consolidation pressure between 25 and 600 psf. Using the new optimal estimation via nonlinear least square approach, the result is around 2.0 ft^2/min under vertical consolidation pressure of 100 psf in the Site 1. Regarding the computation time, the elapsed time from simulations will take place in a few seconds, much shorter than that of the graphical approach being manually implemented. From this preliminary study, it has been shown that the nonlinear modeling and optimization approach could be applied to the consolidation property study successfully.

Conclusions

In this research, some laboratory tests have been initially conducted to determine a number of physical indices and consolidation properties of several composite samples being homogenized from tube samples collected along the coastal Louisiana. The results indicate that the mixed soil samples contain 2.5-15% organic content, which illustrates that the samples are all well degraded. Then low-pressure self-weight consolidation tests are also conducted with the vertical stresses ranging from 10, 25, 50, 100, 200, to 600 psf, which has shown that the coefficient of consolidation tends to decrease with additional pressure. The Log time method yields slightly different values from the root time method. Both methods have caused the variability in the coefficient of consolidation measurement due to the insufficient time for secondary consolidation. On the other hand, fundamental laboratory works involved are labor-intensive. In this case, modeling and optimization approach has been carried out. The hyperbolic tangent function has been introduced to model the curve in the Casagrande method, where nonlinear least square estimation is performed for parameterization optimization. Based on the numerical simulations, the optimal elapse time (t_{50}) and other two parameters are reached instantly, so as to determine the coefficient of consolidation (C_V) together using the empirical equation right away. Outcomes are shown to be similar to those from the Casagrande method and Taylor method.

References

[1] H. Yin, P. Carriere, H. Lawson, H. Mohamadian, and Z. Ye, "Characterization of Salinity Impact on Synthetic Floc Strength Via Nonlinear Component Analysis", Information Management and Big Data: Communications in Computer and Information Science Book Series, V1070, Chapter 2, pp. 8-20, Springer International Publishing, Springer, Switzerland, April 2020

[2] Coastal Wetlands Planning, Protection, and Restoration Act (CWPPRA), 19th Priority Project List Nominee Factsheets and Maps, Louisiana Coastal Wetlands Conservation and Restoration Task Force, USA, 2009

[3] L. Li, I. Alvarez and J. Aubertin, "Self-Weight Consolidation of Slurried Deposition: Tests and Interpretation", International Journal of Geotechnical Engineering 7, 205–213, 2013.

[4] American Society for Testing and Materials (ASTM), "Annual Book of ASTM Standards", US Army Corps of Engineers (USACE), 04-08, 2006

[5] Engineer Manual. No. 1110-2-5027. Engineering and Design, "Confined Disposal of Dredged Material", US Army Corps of Engineers (USACE) Publications, 1987

[6] R. C. Pedersen, "Model Offshore Soil Deposit: Design, Preparation, and Characterization", University of Texas at Austin, 2001

[7] R. G. Robinson, and M. A. Mehter, "Determination of Coefficient of Consolidation from Early Stage of LogT Plot", Geotechnical Testing Journal, Volume 19, Issue 3, September 1996

[8] R. G. Robinson, and M. A. Mehter, "Effect of Clay Mineralogy on Coefficient of Consolidation", Clays and Clay Minerals 46(5):596-600, October 1998

[9] Agus Setyo Muntohar, "Reliability of Method for Determination of Coefficient of Consolidation", pp. 70–75, Proceeding of 13rd Annual Scientific Meeting of Indonesian Society for Geotechnical Engineering (PIT HATTI 13), November 5-6, 2009, Denpasar, Bali

[10] Z. Ye and H. Mohamadian, "Nonlinear Least Squares Regression Analysis and Diagnosis for Optical and Laser Measurement Systems", International Journal of Electrical, Electronics and Data Communication, pp. 1-6, Volume 3, Issue 7, July 2015

[11] Z. Ye, "A Simple Linear Approach for Transient Fuel Control", SAE Technical Paper Series 2003-01-0360, March 3-6, 2003, SAE World Congress, Detroit, USA