Partial-State Feedback Control and Trajectory Specification for a Propeller-Driven Fixed-wing Aircraft

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Abstract—The development of a control for a small RC Aircraft, a small Wilga 2000, is presented. The aircraft dynamics is modeled using Spatial Operator Algebra (SOA) that allows to write the Newton-Euler dynamics equations as a single equation. The propeller, usually modeled by its angular momentum, in this work modeled using SOA and is considered as an independent dynamic body and its effects combined with those of the aircraft, using Newton’s third law, linking the aircraft longitudinal and the lateral-directional models. The model includes the dynamics and the kinematics of the aircraft. For the attitude representation, quaternions are used; while for the control design modified Rodrigues parameters are used. The aircraft has four controls, therefore it is an under-actuated system. A Lyapunov function is proposed to stabilize the attitude position and speed with the body longitudinal axis (x-axis) speed. The trajectory is specified in an additive way by a sequence of attitude and speed changes.

I. INTRODUCTION

There is interest in the control of unmanned autonomous vehicles, fixed or rotary-wing aircrafts among them. This has been made possible thanks to the miniaturization of sensors and computers. Teleoperation allows today the control of an aircraft on the other side of the world. Fixed-wing aircraft have the advantage that they can be designed to be stable within the limits of their flight envelope and require much less energy than a rotary-wing vehicle. Yet they remain, like most vehicles, underactuated systems, moving in a six-dimensional space with four control inputs.

The development of automatic flight controls started with Sperry [1] before the first world war. More recent contributions to flight control design have been made by [2], [3], [4], [5], [6], [7], [8].

Vukobratovic in [2] [3] considers decoupling the system in subsystems beyond the usual longitudinal and lateral-directional system decoupling. A local control is chosen to minimize a local quadratic criterion for a linear model. The methods have been applied successfully to robotic mechanisms and consider the system under a complete integrated dynamic control. [4] in his work uses the stability axes and the nonlinear inverse control laws that use high order derivatives of the state variables, e.g., acceleration, jerk and snap of the position. [5] presents a full envelope longitudinal axis robust control design for a fighter aircraft capable of thrust vectoring. A minimal-order $H_\infty$ is part of the control. [6] presents a three-layer autonomous flight control composite nonlinear law for a small helicopter. [7] presents a passification based robust flight control. [8] considers the control of axisymmetric aerial vehicles where the thrust direction is used to control the system.

In the literature, aircraft are usually modeled as a rigid body using the Newton-Euler methodology [9], [10], [11], [12]. In this paper, the Newton-Euler methodology is used, under the Spatial Operator Algebra formulation [13], similar to Featherstone’s Spatial Vectors Algebra [14], both an evolution of Screw Theory [15]. It has several advantages, it integrates the translation and the rotation on the same frame of reference using a six-dimension vector, it simplifies the modeling, simulation, in this work only the body frame of reference is used. The aerodynamic forces and moments are derived from the work of [16] [10] [12].

The control of under-actuated systems has been studied by [17], a graphic description of the relation (relative degree) between state variables and input variables is given by the control flow diagram (CFD), it proposes a backstepping methodology for the design of the control of systems with triangular form. From their point of view, a fixed-wing aircraft has a cascade structure as it can be seen in figure 2. An important result for this kind of systems is found in the work of M. Spong [18] regarding the partial linearization of under-actuated systems. [19] proposes several normal forms for under-actuated systems; for an aircraft it considers that it has a strict feedback normal form and that the control can be designed using backstepping. [20] reviews different controls schemes for under-actuated system, the Lagrangian method is used to derive the motion equations and extends the classification criteria for under-actuated systems. [21] consist of a compilation of under-actuated systems, their analysis and control; among them, a two-link flying robot is found.

In this work, SOA is used to model a propeller-driven fixed-wing aircraft, a partial feedback linearization is used to stabilize it in attitude and x-axis speed. Lyapunov is used to design the control law and assure its asymptotic stability using.
A peculiarity of the control is that it uses modified Rodrigues parameters for the attitude representation [23] [24]. This representation has a singularity for rotations of $2\pi$ rad, but the control can always use the representation inside de unit circle (short rotation $\leq \pi$ rad) avoiding the singularity.

Finally, an additive scheme for trajectory specification is proposed.

The plane considered for this work is a Wilga 2000 RC aircraft shown in figure 1. To include the propeller dynamics, the paths start at the states diagonal and end at a positive number. A positive number (their weight) and end at the states diagonal, while vertical paths start at the states diagonal and end at a positive number.

![Fig. 1. The model data is from the Graupner Wilga 2000 RC aircraft shown](image)

**II. SYSTEM MODEL**

A. Dynamic Model

The plane considered for this work is a Wilga 2000 RC shown in figure 1. To include the propeller dynamics, the aircraft is modeled as two coupled rigid bodies, using Newton’s thirds law. The aircraft (minus the propeller) dynamic equation, written using spatial operator algebra [13], has the form

$$\mathcal{M}(k)\dot{\mathbf{B}}_g(k) + \mathbf{C}(k) = \mathbf{F}(k)$$

where $k$ is the point where the origin of the frame of reference $\mathcal{B}$, associated with the aircraft body, is located. Its translation and rotation speed $\dot{\mathbf{B}}_g(k) = \mathbf{C}(k)$, is expressed by

$$\mathbf{C}(k) = \{p, q, r, u, v, w\}^T.$$ This is a consistent way of describing rotation and translation of the body by a single equation, i.e. the Newton-Euler dynamic equation, respect to any point of the body. From equation 1, $\mathcal{M}(k)$ is the spatial inertia of the body is given by

$$\mathcal{M}(k) =
\begin{bmatrix}
l_{xx} & 0 & -l_{zc} & 0 & -m_{cg} & m_{cg} \\
0 & I_{yy} & 0 & m_{cg} & 0 & -m_{cg} \\
-l_{zc} & 0 & l_{zz} & -m_{cg} & m_{cg} & 0 \\
0 & m_{cg} & -m_{cg} & m_{cg} & 0 & 0 \\
-m_{cg} & 0 & m_{cg} & 0 & m & 0 \\
m_{cg} & -m_{cg} & 0 & 0 & 0 & m
\end{bmatrix}$$

with $m$ the body mass, $m\{x,y,z\}_{cg}$ the mass multiplied by its position with respect $k$ and $I_x, I_y, I_z \ldots$ the elements of the tensor of inertia with respect $k$.

The external spatial forces $\mathbf{F}(k)\in \mathbb{R}^6$ (union of torques and forces) considered in the model are the gravity $\mathbf{F}_g(k)$, the aerodynamic forces $\mathbf{F}_a(k)$, the propeller forces $\mathbf{F}_p(k)$ and the control forces $\mathbf{F}_u(k)$ due to the elevator, ailerons and rudder, projected in the body frame

$$\mathbf{F}(k) = \mathbf{F}_g(k) + \mathbf{F}_a(k) + \mathbf{F}_p(k) + \mathbf{F}_u(k)U$$

the control forces due to the controls surfaces has the structure

$$\mathbf{F}_u(k) =
\begin{bmatrix}
0 & L_{\delta a} & L_{\delta r} \\
M_{\delta e} & 0 & 0 \\
0 & N_{\delta a} & N_{\delta r} \\
X_{\delta e} & 0 & 0 \\
0 & Y_{\delta a} & Y_{\delta r} \\
Z_{\delta e} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\delta_e \\
\delta_a \\
\delta_r
\end{bmatrix}$$

where the control action principal function is in bold with its secondary, not always desired, consequences in normal font. It can be seen that the coupling between the longitudinal and the lateral-directional models is due to the propeller.

The complete model includes the coupling between the longitudinal and the lateral-directional models is due to the propeller.

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$$\mathcal{M}(k)$$

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**B. Propeller Dynamics**

In the aircraft dynamics literature [11], [12] propellers are included in the Euler equation as objects with angular momentum, or represented by dimensionless coefficients [10]. Here we present an alternative way of including the effects of a rotating body that it is integrated seamlessly to the dynamic equation. Its study arose from the control problems we had during the aircraft take-off.

The propeller dynamic equation is similar to the aircraft body equation, and is given by

\[
\mathcal{M}(p)\dot{\Phi}_B(p) + \mathcal{Y}(p)\mathcal{M}(p)\mathcal{Y}(p) = \mathcal{F}(p)
\]

in this case the external forces: the gravity, the aerodynamic forces and the body force, where the propeller speed (considered fixed to the aircraft) is

\[
\mathcal{Y}(p) = \Phi^*(k, p)\mathcal{Y}(k) + \Omega
\]

where

\[
\Phi(k, p) = \begin{bmatrix}
1 & [k, p] \\
\mathcal{O} & \mathbb{I}
\end{bmatrix} \in \mathbb{R}^{6 \times 6}
\]

is the rigid body operator, where \([k, p]\) in the vector from point \(k\) to point \(p\).

The propeller external forces projected on the propeller frame of reference are

\[
\mathcal{F}(p) = p R_n \mathcal{F}_g \quad + \quad \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
m\gamma
\end{bmatrix} \mathcal{F}_c + n^2 \mathcal{F}_g
\]

therefore the aircraft force on the propeller \(\mathcal{F}_c(p)\) is given by

\[
\mathcal{F}_c(p) = \mathcal{M}(p)\dot{\Phi}_B(p) + \mathcal{Y}(p)\mathcal{M}(p)\mathcal{Y}(p) - \mathcal{F}_g(p) - \mathcal{F}_p(p)
\]

considering Newton’s third law, the force due to the propeller acting on the aircraft is

\[
\mathcal{F}_p(k) = -\Phi(k, p)\mathcal{F}_c(p)
\]

\[
= -\Phi(k, p) \left( \mathcal{M}(p)\dot{\Phi}_B(p) + \mathcal{Y}(p)\mathcal{M}(p)\mathcal{Y}(p) - \mathcal{F}_g(p) - \mathcal{F}_p(p) \right)
\]

the effect of the propeller contact force on the aircraft reference point \(k\) depends on its position \(p\) on the aircraft

\[
\mathcal{F}_p(k) = -\Phi(k, p)\mathcal{M}(p)\Phi^*(k, p)\dot{\Phi}_B(k) - \Phi(k, p)(\Phi^*(k, p)\mathcal{Y}(k) + \Omega)\mathcal{M}(p)(\Phi^*(k, p)\mathcal{Y}(k) + \Omega) + \Phi(k, p)\mathcal{F}_g(p) + \Phi(k, p)\mathcal{F}_p(p)
\]

the system aircraft+propeller dynamics becomes

\[
\begin{align*}
\mathcal{M}(k) + \Phi(k, p)\mathcal{M}(p)\Phi^*(k, p)\dot{\Phi}_B(k) + \\
\mathcal{Y}(k)\mathcal{M}(k)\mathcal{Y}(k) + \\
\Phi(k, p)(\Phi^*(k, p)\mathcal{Y}(k) + \Omega)\mathcal{M}(p)(\Phi^*(k, p)\mathcal{Y}(k) + \Omega) + \\
\Phi(k, p)\mathcal{F}_g(p) + \Phi(k, p)\mathcal{F}_p(p)
\end{align*}
\]

neglecting the relative acceleration of the propeller with respect the aircraft body.

The system aircraft+propeller dynamics can be computed from

\[
\begin{align*}
M(k) &= \mathcal{M}(k) + \Phi(k, p)\mathcal{M}(p)\Phi^*(k, p) \\
\mathcal{Y}(k) &= \Phi^*(k, p)\mathcal{Y}(k) + \Omega \\
M(k)\dot{\Phi}_B(k) + \\
\mathcal{Y}(k)\mathcal{M}(k)\mathcal{Y}(k) + \Phi(k, p)(\Phi^*(k, p)\mathcal{Y}(k) + \Omega)\mathcal{M}(p)(\Phi^*(k, p)\mathcal{Y}(k) + \Omega) + \\
\Phi(k, p)\mathcal{F}_g(k) + \Phi(k, p)\mathcal{F}_a(k) + \Phi(k, p)\mathcal{F}_p(p)
\end{align*}
\]

this model considers the projection of the speeds on the body frame, so \(M(k)\) is constant.

**III. CONTROL DEVELOPMENT**

Our interest on under-actuated or super articulated systems is high because they include a big portion of terrestrial, aerial or nautical robotic systems. From the model, [17] proposed the control flow diagram that shows the relative degree between the control and the state. Figure 2 presents a matrix representation of the diagram for level flight. The system has a cascade structure, since the control interacts simultaneously with all the state variables. The control path flows through the horizontal lines from the controls on the right to the state variables diagonal on the left. The sum of the weight of the path (at the start of the horizontal lines) between the control and the diagonal represents the relative degree between the control and the state variable. As it can be seen from the matrix that the relative degree of all the states of the system \(\{p, q, r, u, v, w, \phi, \theta, \psi, x, y, z\}\) is two.

The direct dynamic model is control affine

\[
\begin{align*}
\dot{\Phi}_B(k) &= M(k)^{-1} \\
&= \begin{bmatrix}
-\mathcal{Y}(k)\mathcal{M}(k)\mathcal{Y}(k) \\
-\mathcal{F}_g(k) - \mathcal{F}_a(k) + \mathcal{F}_p(p) + \mathcal{F}(p)
\end{bmatrix}
\end{align*}
\]

where

\[
\begin{align*}
\mathcal{F} &= M(k)^{-1} \left( -\mathcal{Y}(k)\mathcal{M}(k)\mathcal{Y}(k) + \mathcal{F}_g(k) + \mathcal{F}_a(k) \right) \\
\mathcal{G} &= M(k)^{-1} \mathcal{F}_a(k)
\end{align*}
\]

The Lyapunov function candidate proposed is

\[
V(\delta u, \omega, \sigma) = \frac{1}{2} \begin{bmatrix} \omega \\ \delta u \end{bmatrix} \begin{bmatrix} \omega \\ \delta u \end{bmatrix} + 2K_\sigma \log(1 + \sigma^2) \quad \text{with} \quad |\sigma| \leq 1
\]
it is function of the longitudinal axis (x-axis) speed \( \delta u = u - u_r \), angulars speeds \( \dot{\omega} = (p, q, r) \) and attitude error \( \sigma \), expressed in modified Rodrigues parameters [24], constrained to the unit sphere. Where
\[
\sigma = \left\{ b, b_2, b_3 \right\} = \{\hat{u}_x, \hat{u}_y, \hat{u}_z\} \tan \theta \quad \text{with} \quad \tilde{\sigma} = -\frac{\sigma}{\sigma^2}
\]
the \( \tilde{\sigma} \) is used if \( \sigma \) exits the unit circle (equivalent of choosing the short rotation).

The time derivative of the Lyapunov function candidate is given by
\[
V = \left[ \omega \quad \delta u \right] \left[ \frac{\omega}{\delta u} \right] + K_d \sigma^T \omega \quad \text{(24)}
\]
\[
= \left[ \omega \quad \delta u \right] \left( F_4 + G_4 U + K_p \left[ \sigma \right] \right) \quad \text{(25)}
\]
\[
= \left[ \omega \quad \delta u \right] \left( \omega \quad \delta \dot{u} \right) \quad \text{(26)}
\]
\[
= -\left[ \omega \quad \delta \dot{u} \right] K_d \left[ \omega \quad \delta \dot{u} \right] < 0 \quad \text{(27)}
\]
therefore the stabilizing control is given by
\[
U = G_4^{-1} \left( -F_4 - K_d \left[ \omega \right] - K_p \left[ \sigma \right] \right) \quad \text{(28)}
\]
this only assures the stability for the controlled speeds. \( G_4 \) relates the control surface movements and the spatial forces due to them and will have an inverse in the controllable speed range. The system behaviour under this control is given by the direct dynamic equation
\[
\dot{\beta}_n (k) = F + GU \quad \text{(29)}
\]
\[
= F + G G_4^{-1} \left( -F_4 - K_p \left[ \sigma \right] - K_d \left[ \omega \right] \right) \quad \text{(30)}
\]
this splits the system into a directly and indirectly controlled subsystems that corresponds to a collocated linearization of the system [18], with \( \dot{\beta}_{B4} (k) = \{ p, q, r, u \} \) and \( \dot{\beta}_{B2} (k) = \{ v, w \} \)
\[
\dot{\beta}_{B4} (k) = -K_p \left[ \sigma \right] - K_d \left[ \omega \right] \quad \text{(31)}
\]
\[
\dot{\beta}_{B2} (k) = F_2 + G_2 G_4^{-1} \left( -F_4 + \dot{\beta}_{B4} (k) \right) \quad \text{(32)}
\]
these are the equations of the controlled system.

In order to have asymptotic stability the result on [22] is applied. Since we only have
\[
\dot{V} (\omega, \delta u, \sigma) = -\left[ \omega \quad \delta u \right] K_d \left[ \omega \quad \delta \dot{u} \right] < 0 \quad \text{(33)}
\]
the evaluation on the null speed of subsequent time derivatives of the Lyapunov function give us
\[
\ddot{V} (\omega, \delta u, \sigma) = -2 \left[ \omega \quad \delta u \right] K_d \left[ \omega \quad \delta \dot{u} \right] = 0 \quad \text{(34)}
\]
therefore we have a asymptotically stable control.

IV. TRAJECTORY SPECIFICATION

The trajectory is specified as a set of attitude and speed changes. The basic trajectory change involves change in glide angle, direction, and speed.
\[
\delta T = \{ t_{start}, \delta \theta, \delta \psi, \delta V \} \quad \text{(39)}
\]
so a trajectory is a sequence given by
\[
\{ t_1, \delta \theta_1, \delta \psi_1, \delta V_1 \}
\]
\[
\{ t_2, \delta \theta_2, \delta \psi_2, \delta V_2 \}
\]
\[
\ldots
\]
\[
\{ t_n, \delta \theta_n, \delta \psi_n, \delta V_n \}
\]
with \( t_{i+1} > t_i + \delta_t \)
where \( \psi_n = \psi_0 + \sum_{i=1}^{n} \delta \psi_i \)
\[
\theta_n = \theta_0 + \sum_{i=1}^{n} \delta \theta_i
\]
\[
V_n = V_0 + \sum_{i=1}^{n} \delta V_i
\]
from a starting state \( \{ \theta_0, \psi_0, V_0 \} \).

V. NUMERIC EXPERIMENT

The previous models were programmed in Mathematica and the result of a numeric experiment is presented here. The experiment shows the advantages and limitations of the control. The model uses quaternions to represent the attitude. However, the results are presented using Euler angles, since they are easier to interpret.

The experiment consisted of a horizontal flight segment followed by a climb, a couple of right turns followed by a couple of left turns and a descent. The \( u \) speed is constant during most of the flight, only during the descent is increased. The vehicle state, control and trajectory evolution are shown in figures 3, 4 and 5. The control was tuned with two identical negative real roots (-2.5, -2.5).
VI. CONCLUSIONS AND FUTURE WORK

A. Conclusions

A fixed-wing aircraft was modelled with the help of spatial operator algebra, on a flat-earth. The propeller was modelled with the same tool as a system (with one degree of freedom) in contact with the aircraft. Both models were integrated into a single model. The model was used for numeric experiments presented in this work. Quaternions were used to represent the system attitude. This work follows the work presented in [25].

The system considered is under-actuated. The control flow diagram for this system seems a cascade system, so with the control branches associated with the attitude and the linear speed on the body x-axis, the rest of the variables is controlled. The stability control was developed using a Lyapunov function. The control uses the inverse dynamic model that should exist beyond the stall speed. The control was developed using a Lyapunov function. The control uses the inverse dynamic model that should exist beyond the stall speed. The control is tuned as a critical damped linear system.

B. Future Work

The control works on the body frame of reference. Yet the attitude is given in the navigation frame. An outer control loop is needed to control the altitude and make coordinate turns possible.

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