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Generalized Admittance Matrix Model of Transformers

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***Abstract*—This brief describes a new admittance matrix model to approach the behavior of transformers. The new model is simpler than the one reported in the literature and can directly be used to fill the admittance matrix without extra variables. Our results indicate that the proposed stamp can directly be used into a standard nodal analysis, obtaining a reduced and sparse system of equations. Two analysis examples are provided, showing that fully-symbolic transfer functions of electronic circuits based on transformers can be computed by using nodal analysis only. As a consequence, the computational complexity during the solution of the system of equations is reduced when recursive determinant expansion techniques are applied.**

***Index Terms*—Nodal analysis, symbolic analysis, transformers, controlled sources, nullor.**

I. Introduction

Analysis of electrical networks is the base of any circuit simulator or symbolic analyzer in order to obtain the numerical or symbolic behavioral model of the electrical network [1]–[4]. Among all the electrical network analysis techniques reported in the literature, the modified nodal analysis (MNA) method has widely been used in numerical simulators or symbolic analyzers in order to formulate the system of equations, where the element stamp method is used to fill the admittance matrix. In this sense, controlled sources are used to model the behavior of active devices. However, the traditional stamp associated to the voltage-controlled voltage source (VCVS) and current-controlled voltage source (CCVS) has a negative impact in the filling of the admittance matrix, since additional rows and columns together with nonzero elements must be added. As consequence, the MNA matrix not only becomes large and dense, but the computational complexity increases when recursive determinant-expansion techniques are applied to compute fully-symbolic small-signal characteristics of electrical networks [1]–[4]. This shortcoming has partially been overcome by the deduction of the element stamp for the two transactors mentioned above or by modeling each of them with pathological elements [5]–[9]. However, the element stamp of transformers has not been deduced until today, in order to be used directly in a nodal analysis (NA) [1]–[4]. Therefore, as an extension of [10]–[12], a generalized admittance matrix model for transformers is herein introduced and it can be used in a standard NA for generating fully-symbolic small-signal characteristics of electric circuits.

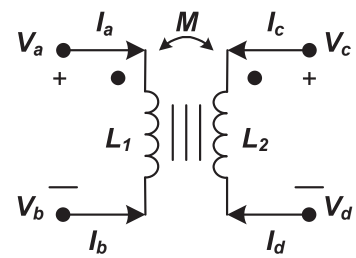

Fig. 1. Transformer with positive couplig M.

The paper is structured as follows: In Section II, the NA and MNA formulation methods along with the traditional stamp of a transformer are briefly revised. In Section III, the derivation of the admittance matrix model for the transformer is introduced. The compute of fully symbolic transfer functions of electrical networks based on transformers are introduced in Section IV. Finally, conclusions are summarized in Section V.

II. Traditional Element Stamp of a Transformer

NA is a classical formulation method that offers low programming effort, low storage requirements, efficient set-up time and high execution speed. NA is represented by

$$Y_n V = J \tag{1}$$

where J is the current source vector, Y_n represents the admittance matrix and V is the node voltage vector. Although the element stamp method is used to fill (1), the behavior of transformers and independent voltage sources can not be directly included and hence, the MNA method must still be used. The MNA matrix is expressed as

$$\begin{bmatrix} Y_n & B \\ C & D \end{bmatrix} \begin{bmatrix} V \\ I \end{bmatrix} = \begin{bmatrix} J \\ E \end{bmatrix} \tag{2}$$

where B, C and D correspond to branch equations for the branches whose currents are in the branch current vector I and E is the independent voltage vector. It is worth mentioning that the original stamp for each non-NA compatible element was deduced to fill (2) [1]–[4]. Later on, improved versions were developed to fill (1) instead (2). In this way, the improved stamp for VCVS can be found in [9]–[12] and the admittance matrix stamp for CCVS can be found in [5], [10], [11]. From the improved stamps, nonlinear macromodels of operational amplifiers [13], current feedback operational amplifiers [14] and current conveyors were also developed [15], [16]. For the case of the independent voltage source, a Norton equivalent is applied to transform this source to independent current source by using its own source resistance. As a consequence, the admittance matrix is reduced in one order. However, the transformer shown in Fig. 1, cannot be directly included in (1), since its stamp is given by

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ m+1 \\ m+2 \end{array}\begin{array}{c} \begin{array}{cccccc} a & b & c & d & I_a & I_c \end{array} \\ \begin{bmatrix} & & & & 1 & \\ & & & & -1 & \\ & & & & & 1 \\ & & & & & -1 \\ 1 & -1 & & & -sL_1 & -sM \\ & & 1 & -1 & -sM & -sL_2 \end{bmatrix} \end{array} \quad (3)$$

which is more suitable for use in (2). In (3), L_1 and L_2 are the self inductances for primary and secondary windings and M is the mutual inductance between the two inductors.

III. Element Stamp for Transformers

Transformers are significant building blocks in the low- and high-frequency analog signal processing. For instance, in the design of some radio frequency power amplifiers, transformers are used as active load or impedance matching [17]. However, from point of view of computer aided circuit analysis, the number of nonzero elements along with the size of the MNA matrix increases whether the stamp given in (3) is used by each transformer. This is a serious drawback, since the computational complexity increases when the determinant of large matrices is computed by using recursive determinant-expansion techniques. In front of this situation, the goal of this section is to deduce the NA stamp of Fig. 1. Simple analysis of Fig. 1 allows obtaining a set of equations given by

$$\begin{aligned} V_a - V_b &= sL_1 I_a + sM I_c \\ V_c - V_d &= sM I_a + sL_2 I_c \\ I_c &= -I_d \\ I_a &= -I_b \end{aligned} \quad (4)$$

Re-writing (4) in terms of nodal voltages, the generalized admittance matrix is deduced as

$$\frac{\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} L_2 & -L_2 & -M & M \\ -L_2 & L_2 & M & -M \\ -M & M & L_1 & -L_1 \\ M & -M & -L_1 & L_1 \end{bmatrix} \end{array}}{s(L_1L2-M^2)} \begin{bmatrix} V_a \\ V_b \\ V_c \\ V_d \end{bmatrix} = \begin{bmatrix} I_a \\ I_b \\ I_c \\ I_d \end{bmatrix} \quad (5)$$

Note that if M=0, i.e., the magnetic coupling is null, then (5) is reduced to

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} \frac{1}{sL_1} & -\frac{1}{sL_1} & 0 & 0 \\ -\frac{1}{sL_1} & \frac{1}{sL_1} & 0 & 0 \\ 0 & 0 & \frac{1}{sL_2} & -\frac{1}{sL_2} \\ 0 & 0 & -\frac{1}{sL_2} & \frac{1}{sL_2} \end{bmatrix} \end{array} \begin{bmatrix} V_a \\ V_b \\ V_c \\ V_d \end{bmatrix} = \begin{bmatrix} I_a \\ I_b \\ I_c \\ I_d \end{bmatrix} \quad (6)$$

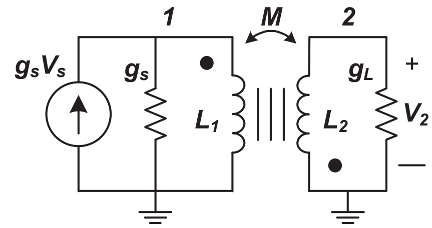

Fig. 2. Simple circuit based on transformer with negative couplig M.

which describes the associated stamp to each inductor L_1 and L_2. From (5), one can observe that sixteen nonzero elements form the admittance matrix whereas (3) has only twelve nonzero elements. However, the order of (5) is lower than the order of (3) and as a consequence, (5) can directly be used in (1), saving computing resources.

IV. Illustrative Examples

Let us consider the simple circuit depicted in Fig. 2 in order to show the potentiality of the proposed stamp, where the independent voltage source has been transformed to independent current source by applying a Norton equivalent. According to Fig. 2, the stamp given by (3) is applied to fill the admittance matrix given by

$$\begin{bmatrix} g_s & 0 & 1 & 0 \\ 0 & g_L & 0 & -1 \\ 1 & 0 & -sL_1 & -sM \\ 0 & -1 & -sM & -sL_2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ I_a \\ I_d \end{bmatrix} = \begin{bmatrix} g_sV_s \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (7)$$

Now if the proposed stamp is used, the admittance matrix is obtained as

$$\begin{bmatrix} g_s + \frac{L_2}{s(L_1L_2-M^2)} & \frac{M}{s(L_1L_2-M^2)} \\ \frac{M}{s(L_1L_2-M^2)} & g_L + \frac{L_1}{s(L_1L_2-M^2)} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} g_sV_s \\ 0 \end{bmatrix} \quad (8)$$

Comparing (7) and (8), it can be seen that the size of the former is 4×4 with 10 nonzero elements whereas the latter has 4 nonzero elements and the order is 2×2. Hence, the computational complexity will be reduced when recursive determinant-expansion techniques are applied to solve (8) instead of (7). For each case, the output voltage is taken at V_2 and given by

$$\frac{V_2(s)}{V_s(s)} = -\frac{sg_sM}{(L_1L_2g_Lg_s - M^2g_Lg_s)s^2 + (L_1g_s + L_2g_L)s + 1} \quad (9)$$

As second example, let us consider the tuned double power amplifier shown in Fig. 3(a). The amplifier has two tuned circuits L_1C_1 and L_2C_2 where the output signal of the former tuned circuit is coupled to the second through mutual coupling. The small-signal equivalent circuit is illustrated in Fig. 3(b), where a Norton equivalent has been applied in the independent voltage source in order to be transformed to current source and g_{eq}=$g_a + g_b$. According to Fig. 3(b), the stamps given by (3) and (5) are applied to fill the MNA and NA matrices, respectively, which are given by (10) and (11). Again, comparing these systems of equations, one can note that the size of the former is 7×7 with 18 nonzero elements and (11)

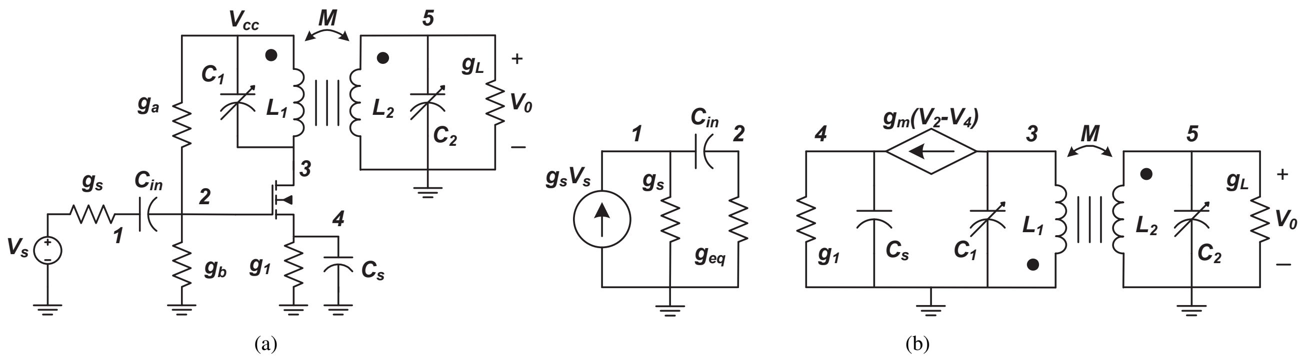

Fig. 3. (a) Tuned double power amplifier, (b) Small-signal equivalent circuit.

$$\begin{bmatrix} g_s+sC_{in} & -sC_{in} & 0 & 0 & 0 & 0 & 0 \\ -sC_{in} & g_{eq}+sC_{in} & 0 & 0 & 0 & 0 & 0 \\ 0 & g_m & sC_1 & -g_m & 0 & -1 & 0 \\ 0 & -g_m & 0 & g_m+g_1+sC_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g_L+sC_2 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & -sL_1 & -sM \\ 0 & 0 & 0 & 0 & 1 & -sM & -sL_2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ I_b \\ I_c \end{bmatrix} = \begin{bmatrix} g_sV_s \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{10}$$

$$\begin{bmatrix} g_s+sC_{in} & -sC_{in} & 0 & 0 & 0 \\ -sC_{in} & g_{eq}+sC_{in} & 0 & 0 & 0 \\ 0 & g_m & sC_1+\frac{L_2}{s(L_1L_2-M^2)} & -g_m & \frac{M}{s(L_1L_2-M^2)} \\ 0 & -g_m & 0 & g_m+g_1+sC_s & 0 \\ 0 & 0 & \frac{M}{s(L_1L_2-M^2)} & 0 & g_L+sC_2+\frac{L_1}{s(L_1L_2-M^2)} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix} = \begin{bmatrix} g_sV_s \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{11}$$

$$\frac{V_5(s)}{V_s(s)} = \frac{C_{in}g_mMg_s(C_ss+g_1)s^2}{(1+as^4+bs^3+cs^2+L_2g_Ls)(C_ss+g_1+g_m)(C_{in}(g_{eq}+g_s)s+g_{eq}g_s)} \tag{12}$$

has 12 nonzero elements and the size of the admittance matrix is 5×5. Therefore, one can insight that the computational complexity is also reduced. On the other hand, the output voltage is found in V_5 for both admittance matrices and given by (12) without loss accuracy, where a=$C_1C_2L_1L_2 - C_1C_2M^2$, b=$C_1L_1L_2g_L - C_1M^2g_L$ and c=$C_1L_1 + C_2L_2$. Therefore, fully-symbolic small-signal characteristics of analog circuits containing transformers can easily be computed by applying a standard NA, which means that the computational complexity and memory consumption during the solution of the system of equations by using recursive determinant expansion techniques are improved. It is worth mentioning that the parasitic capacitances C_{gs} and C_{gd} along with the parasitic resistance g_0 associated to the MOSFET of Fig. 3(a) were not considered. It is because when the fully-symbolic transfer functions were computed considering these parasitic elements, a large quantity of symbolic sum-of-products were generated. Therefore, terms reduction algorithms are still required in order to obtain symbolic transfer functions with dominant terms only [3], [4]. Finally, whereas Table I summarizes the size of the admittance matrix and the generation of nonzero coefficients according to the formulation method, Table II shows the average CPU-time and memory consumption required to solve the systems of equations, by applying Laplace expansion method on a 2-GHz Intel Cored 2 Duo machine with 2GB RAM. As one can observe, less CPU-time and memory consumption are required to solve (7) and (10) instead of (8) and (11).

TABLE I
COMPARISON OF FORMULATION METHODS

Formulation method	Size		Nonzero coefficients	
	Fig. 2	Fig. 3(b)	Fig. 2	Fig. 3(b)
MNA	4×4 (7)	7×7 (10)	10	18
NA	2×2 (8)	5×5 (11)	4	12

TABLE II
CPU-TIME AND MEMORY USE

Equation	CPU-time (ms)	Memory (bytes)
(7)	0.12	1,635
(8)	0.03	739
(10)	0.15	2,933
(11)	0.06	1,034

V. CONCLUSIONS

A new generalized admittance matrix model for transformers has been proposed. Leveraging this new stamp, fully-symbolic transfer functions of electrical networks based on transformers were computed by using NA only. The main advantage of the proposed stamp is that not only small matrices can be formulated compared with the traditional MNA formulation, but CPU-time and memory consumption during the solution process will be improved as has already been shown in [6]. Furthermore, the proposed stamp becomes part of the same set of improved stamps associated to the transactors reported in [5], [9]–[12] and as a consequence, NA can now be done for any electric network.

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