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Trajectory Tracking of Parallel Robots: A Relaxed Differential-Algebraic-Equation Approach

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***Abstract*—In this paper, trajectory tracking for a class of parallel robots is pursued via a novel computed-torque technique; it employs an exact convex representation of the algebraic constraints in order to include them, via the Finsler's Lemma, in the Lyapunov-based analysis of the outer-loop feedback. In contrast with other results that intend to include algebraic constraints for controller design, the resulting conditions can be expressed as linear matrix inequalities via an adequate choice of variables; this characteristic proves useful to directly specify decay rate and input constraints. Conservativeness of former approaches is reduced as it is shown that the proposal includes ordinary proportional-derivative outer-loop feedback as a particular case. The scheme is put at test on a 2-degrees-of-freedom planar parallel manipulator characterized by a set of differential algebraic equations.**

***Index Terms*—Parallel Robots, Computed-Torque Control, Tracking, Differential algebraic equations, Linear Matrix Inequality.**

I. Introduction

Very often, rigid robot manipulators consisting in open kinematic chains –both actuated and underactuated– are modelled as Lagrange-Euler systems by using first principles; these physical laws are based on the arms potential and kinetic energies [1]. This representation is suitable for feedback linearization in order to achieve trajectory tracking [2], where the model inversion is performed by an inner-loop feedback and the actual tracking relies on an outer-loop control, usually designed as a series of decoupled proportional-derivative controllers [3]. These are the basis of what is known as the computed-torque technique, which is the most common approach for driving a manipulator to a desired trajectory [4]; it usually assumes that a proper path has been generated in the robot workspace [5], that enough memory is allocated for the via points [6], and that discretisation issues have been considered, if employed [7]. The success of computed-torque techniques can be measured by the wide variety of real-time applications based on them [8], [9], [10].

When a kinematic chain is closed by joining the end effector to the base, a parallel robot is produced [11]. In such case, the designer willing to perform trajectory tracking faces several new challenges: algebraic restrictions representing the closures must be added to the Lagrange-Euler dynamical equations, thus producing what is known as a set of differential algebraic equations (DAEs); simulation of a DAE requires consistent initialization [12] and the implementation of the Pantelides algorithm to recover missing dynamics [13]; computed-torque techniques cannot be directly applied as possibly conflicting interactions might arise that should be taken into account [14]. While most of the solutions to handle these problems do not consider the DAE nature of the problem [15], [16], some recent works have begun to consider it [17], [18]: this work adopts the latter perspective.

Inspired by the analysis of active and passive joints in [18], where modelling and control of an ankle reeducation device known as the motoBOTTE has been performed based on its DAE representation, this paper proposes a way to incorporate the algebraic restrictions of parallel robots into the outer-loop feedback Lyapunov-based design. Based on the Finsler's Lemma [19], algebraic restrictions inherited from geometrical properties [20] and their time derivatives reducing the degrees of freedom (DOF) [21] are incorporated into the analysis. These restrictions are exactly rewritten as a convex sum of linear expressions, by means of the sector nonlinearity approach [22], which allows using them in Lyapunov analysis to obtain conditions in the form of linear matrix inequalities (LMIs) for the outer-loop feedback [23], [24]. LMIs are highly appreciated for controller design as they can be efficiently solved in polynomial time by commercially available software [25], [26]; moreover, they allow input saturation limits, decay rate, and other performance measures to be easily incorporated [27]. The enhanced control law is then included in the usual computed-torque inner-loop feedback, resulting in a bigger family of solutions that includes the usual approach as a particular case; hence, the relaxation.

Results are presented as follows: section II provides the very basics on ordinary computed-torque technique, DAE formulations for parallel robots, and the Finsler's Lemma; section III shows how the algebraic constraints can be convexly modelled and incorporated into Lyapunov-based analysis, thus proving a theorem that formally establishes the new computed-torque proposal; section IV illustrates the proposed technique in a 2-DOF planar parallel manipulator and discusses the advantages and disadvantages of the approach; conclusions are gathered in section V.

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II. Preliminaries

Since this work is concerned with a relaxed approach of the computed-torque technique for parallel robots, this section presents the standard one for open kinematic chains; then we explain how closed kinematic chains can be modelled by DAEs and what issues do they face for simulation and control; finally, the Finsler's Lemma is presented as a way to introduce algebraic constraints into Lyapunov analysis.

A. The standard computed-torque technique

Rigid-body robotic manipulators consisting of an open chain of links with n joint variables (rotational or prismatic) gathered in a vector $q \in \mathbb{R}^n$, are always amenable to the Lagrange-Euler form

$$M(q)\ddot{q}(t) + V(q,\dot{q}) + F(q,\dot{q}) + G(q) = \tau(t), \qquad (1)$$

where $M(q) \in \mathbb{R}^{n\times n}$, $V(q,\dot{q}) \in \mathbb{R}^n$, $F(q,\dot{q}) \in \mathbb{R}^n$, and $G(q) \in \mathbb{R}^n$, are the inertia matrix, the Coriolis, friction, and gravity vectors, respectively. Torques (for rotational motions) and linear forces (for prismatic displacements) are gathered in the generalized force vector $\tau(t) \in \mathbb{R}^n$. If all the entries in $\tau(t)$ are available for control purposes, i.e., if the plant is fully actuated, a desired trajectory $q_d(t)$ can be asymptotically tracked by $q(t)$ if the control law

$$\tau(t) = M(q)(\ddot{q}_d(t) - u(t)) + V(q,\dot{q}) + F(q,\dot{q}) + G(q) \qquad (2)$$

is applied, with

$$u(t) = -K_p e(t) - K_v \dot{e}(t), \qquad (3)$$

being an outer-loop feedback with $e(t) = q_d(t) - q(t)$, $K_p = \text{diag}\{k_{p1}, k_{p2}, \cdots, k_{pn}\}$, and $K_v = \text{diag}\{k_{v1}, k_{v2}, \cdots, k_{vn}\}$ diagonal matrix gains whose pairs (k_{pi}, k_{vi}), $i \in \{1, 2, \ldots, n\}$, can be established as $k_{pi} = \omega_n^2$ and $k_{vi} = 2\zeta\omega_n =$, with ω_n being the natural frequency and ζ the damping rate required for the i-th joint variable $q_i(t)$ in asymptotically reaching the i-th desired trajectory $q_{di}(t)$ [1].

Standard computed-torque control has the advantage of reducing a tracking task to model inversion for the inner-loop feedback (2) and linear control for the outer-loop one (3); disadvantages include the need of precise knowledge of the model, full availability of actuators (thus, underactuated systems require difficult adaptations [28]), and lack of richness for gains K_p and K_v as they are forced to be only diagonal for simplicity.

B. Parallel robots and differential algebraic equations

Parallel robots are closed kinematic chains; they are usually modelled as follows:

1) Open the kinematic chain at any number of convenient points and model the s different parts as independent dynamics

$$\begin{gathered}
M^1(q^1)\ddot{q}^1(t) + V^1(q^1,\dot{q}^1) + F^1(q^1,\dot{q}^1) + G^1(q^1) = \tau^1(t),\\
M^2(q^2)\ddot{q}^2(t) + V^2(q^2,\dot{q}^2) + F^2(q^2,\dot{q}^2) + G^2(q^2) = \tau^2(t),\\
\vdots\\
M^s(q^s)\ddot{q}^s(t) + V^s(q^s,\dot{q}^s) + F^s(q^s,\dot{q}^s) + G^s(q^s) = \tau^s(t),
\end{gathered}$$

where the superscript stands for indexation.
Note that these equations can be gathered into a single one with the usual Lagrange-Euler structure; indeed, if $q \equiv \begin{bmatrix}(q^1)^T & (q^2)^T & \cdots & (q^s)^T\end{bmatrix}$, then

$$M(q)\ddot{q}(t) + V(q,\dot{q}) + F(q,\dot{q}) + G(q) = \tau(t), \qquad (4)$$

with (omitting arguments)

$$M(q) \equiv \text{block-diag}\{M^1(q^1), M^2(q^2), \ldots, M^s(q^s)\},$$

$$V(q,\dot{q}) = \begin{bmatrix} V^1\\ V^2\\ \vdots\\ V^s\end{bmatrix},\ F(q,\dot{q}) = \begin{bmatrix} F^1\\ F^2\\ \vdots\\ F^s\end{bmatrix},\ G(q) = \begin{bmatrix} G^1\\ G^2\\ \vdots\\ G^s\end{bmatrix}.$$

2) Close the kinematic chain by adding the corresponding geometrical constraints in the form of algebraic equations; they can always be gathered as:

$$R(q)q + r = 0, \qquad (5)$$

where $R(q) \in \mathbb{R}^{m\times n}$ and $r \in \mathbb{R}^m$ describe m algebraic restrictions that give account of the parallel characteristics of the system; these characteristics must be structural.

Note that (4)-(5) is a set of 2nd-order DAEs; this means that the dynamics (4) can only induce motion within the subset of the state-space defined by (5). Needless to say, this requires consistent initialization of states and inputs as they must lie within the restricted space given by (5); failure to meet this requirement for control purposes may lead to impulsive behavior [12].

C. The Finsler's Lemma

The design of the outer-loop feedback (3) in standard computed-torque technique is made by ordinary pole placement [29] in a linear error system derived from substituting (2) in (1). Alternatively, LMI formulations can be applied instead to find a richer set of gains for the outer-loop feedback [27]. LMIs are usually derived by applying the direct Lyapunov method [23]; e.g., for linear systems $\dot{x}(t) = Ax(t)$ the quadratic Lyapunov function candidate $V(x) = x^T P x$, $P > 0$, along with its time derivative $\dot{V} = x^T Q x < 0$, $Q = PA + A^T P$, can be used for stability analysis. Yet, when algebraic restrictions $Rx = 0$ are known to apply, the inequality $\dot{V} = x^T Q x < 0$ is only necessary in the subspace $Rx = 0$. This is precisely the statement of the Finsler's Lemma:

Lemma 1: [30] For $x \in \mathbb{R}^n$, $Q = Q^T \in \mathbb{R}^{n\times n}$, and $R \in \mathbb{R}^{m\times n}$, $\text{rank}(R) < n$:

$$\begin{matrix} x^T Q x < 0, \\ \forall Rx = 0,\, x \neq 0 \end{matrix} \iff \begin{matrix} \exists N \in \mathbb{R}^{n\times m}: \\ Q + NR + R^T N^T < 0. \end{matrix} \qquad (6)$$

In the next section we will develop a methodology where this Lemma will play an essential role in relaxing the computed-torque technique when applied to parallel robots as these exhibit algebraic restrictions that can be taken into account when deriving the outer-loop feedback by Lyapunov methods.

III. Main Results

Consider a parallel robot whose model is the set of 2nd-order DAEs (4)-(5); an error system may arise from two sources:

1) substituting (2) in (4) without any consideration of the restriction (5), which can be incorporated later into the analysis; the size of q remains the same, i.e, $q \in \mathbb{R}^n$, and the error system has the state $\bar{e} = \begin{bmatrix} e^T & \dot{e}^T \end{bmatrix}^T \in \mathbb{R}^{2n}$, with $e(t) = q_d(t) - q(t)$, or
2) considering the restriction (5) and as many derivatives of it as to reduce by substitution the dynamical part (4) to a minimum size whose dynamics are enough to determine the remaining joint variables by the algebraic relationships, i.e.:

$$\left.\begin{array}{c} M(q)\ddot{q}(t) + V(q,\dot{q}) + F(q,\dot{q}) + G(q) = \tau(t) \\ R(q)q + r = 0,\ \dot{R}(q)q + R(q)\dot{q} = 0, \\ \ddot{R}(q)q + 2\dot{R}(q)\dot{q} + R(q)\ddot{q} = 0 \end{array}\right\}$$
$$\Rightarrow \tilde{M}(\tilde{q})\ddot{\tilde{q}}(t) + \tilde{V}(\tilde{q},\dot{\tilde{q}}) + \tilde{F}(\tilde{q},\dot{\tilde{q}}) + \tilde{G}(\tilde{q}) = \tilde{\tau}(t), \quad (7)$$

where $\tilde{q}$ is a vector whose size is necessarily lower than that of q; $\tilde{M}$, $\tilde{V}$, $\tilde{F}$, and $\tilde{G}$ are the resulting matrices of this process. The reduced dynamics also have a Lagrange-Euler form and can therefore be subject to a computed-torque control law of the form (2):

$$\tilde{\tau}(t) = \tilde{M}(\tilde{q})(\ddot{\tilde{q}}_d(t) - u(t)) + \tilde{V}(\tilde{q},\dot{\tilde{q}}) + \tilde{F}(\tilde{q},\dot{\tilde{q}}) + \tilde{G}(\tilde{q}), \quad (8)$$

which, once substituted in (7), produces an error system with the same structure as in the first option; the only difference being the reduced size of the resulting error vector $e(t) = \tilde{q}_d(t) - \tilde{q}(t)$, where $\tilde{q}_d(t)$ is the induced desired trajectory from $q_d(t)$.

Irrespective of the choice above, the resulting error system has always the following structure; keep in mind, though, that the dimension of e is n in the first choice and less than n in the second choice above:

$$\begin{bmatrix} \dot{e}(t) \\ \ddot{e}(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e(t) \\ \dot{e}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u(t), \quad (9)$$

where I denotes an identity matrix of appropriate size.

Moreover, note that the set of restrictions as well as its time derivative can always be written as follows (see factorization in [31], based on an explicit formulation of the differential mean value theorem):

$$R(q_d)q_d + r - (R(q)q + r) = \bar{R}^1(q,q_d)e = 0,$$
$$\dot{R}(q_d)q_d + R(q_d)\dot{q}_d - \left(\dot{R}(q)q + R(q)\dot{q}\right)$$
$$= \begin{bmatrix} \bar{R}^2(q,q_d,\dot{q},\dot{q}_d) & \bar{R}^3(q,q_d,\dot{q},\dot{q}_d) \end{bmatrix} \begin{bmatrix} e(t) \\ \dot{e}(t) \end{bmatrix} = 0,$$

where $\bar{R}^1$, $\bar{R}^2$, and $\bar{R}^3$ are matrices of appropriate dimensions ($m \times n$ in the first choice above to obtain the error system) whose entries are possibly nonlinear expressions of their arguments. These restrictions can be put together as:

$$\begin{bmatrix} \bar{R}^1(q,q_d) & 0 \\ \bar{R}^2(q,q_d,\dot{q},\dot{q}_d) & \bar{R}^3(q,q_d,\dot{q},\dot{q}_d) \end{bmatrix} \begin{bmatrix} e(t) \\ \dot{e}(t) \end{bmatrix} = 0 \quad (10)$$

Consider the compact set $\Omega \subset \mathbb{R}^{4n}$ where q, q_d, $\dot{q}$, and $\dot{q}_d$ lie due to the plant physical constraints; let us assume they induce boundedness in the different non-constant expressions in $\bar{R}^1$, $\bar{R}^2$, and $\bar{R}^3$, listed as $z_1(\bar{q}) \in [z_1^0, z_1^1]$, $z_2(\bar{q}) \in [z_2^0, z_2^1]$, ..., $z_p(\bar{q}) \in [z_p^0, z_p^1]$, where $\bar{q} = \begin{bmatrix} q^T & q_d^T & \dot{q}^T & \dot{q}_d^T \end{bmatrix}^T$. Defining

$$w_0^i(\bar{q}) = \frac{z_i^1 - z_i(\bar{q})}{z_i^1 - z_i^0},\ w_1^i(\bar{q}) = \frac{z_i(\bar{q}) - z_i^0}{z_i^1 - z_i^0}, \quad (11)$$

it can be checked that the non-constant terms can be written as convex sums

$$z_i(\bar{q}) = w_0^i(\bar{q})z_i^0 + w_1^i(\bar{q})z_i^1, \quad (12)$$

where $\forall \bar{q} \in \Omega$, $w_0^i(\bar{q}) + w_1^i(\bar{q}) = 1$, $i \in \{1,2,\ldots,p\}$, $w_j^i(\bar{q}) \in [0,1]$.

Convex sums can be stacked together at the leftmost side of expressions, which implies that restriction (10) can be equivalently written as:

$$\sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\cdots\sum_{i_p=0}^{1} w_{i_1}^1 w_{i_2}^2 \cdots w_{i_p}^p \begin{bmatrix} \bar{R}^1_{(i_1,i_2,\ldots,i_p)} & 0 \\ \bar{R}^2_{(i_1,i_2,\ldots,i_p)} & \bar{R}^3_{(i_1,i_2,\ldots,i_p)} \end{bmatrix} \begin{bmatrix} e \\ \dot{e} \end{bmatrix} = 0,$$

where $\bar{R}^j_{(i_1,i_2,\ldots,i_p)} = \bar{R}^j(\bar{q})|_{w_{i_1}^1 = w_{i_2}^2 = \cdots = w_{i_p}^p = 1}$, $j \in \{1,2,3\}$, are constant matrices. With the aid of notation $\mathbb{B} \equiv \{0,1\}$, $\mathbf{i} \equiv (i_1, i_2, \ldots, i_p)$, and $\mathbf{w_i}(\bar{q}) = w_{i_1}^1(\bar{q}) w_{i_2}^2(\bar{q}) \cdots w_{i_p}^p(\bar{q})$, the nested convex sums above can be further simplified as

$$\sum_{\mathbf{i} \in \mathbb{B}^p} \mathbf{w_i}(\bar{q}) \begin{bmatrix} \bar{R}^1_{\mathbf{i}} & 0 \\ \bar{R}^2_{\mathbf{i}} & \bar{R}^3_{\mathbf{i}} \end{bmatrix} \begin{bmatrix} e \\ \dot{e} \end{bmatrix} = 0. \quad (13)$$

Note that (10) is equivalent to (13); it is not an approximation.

It is now possible to state the main result of this paper:

Theorem 1: The origin of the error system (9) (resulting from substitution of (2) in (4) or (8) in (7)) under the control law (3), where the algebraic restrictions and its derivatives (10) can be rewritten as (13) on the induced compact set $\Omega_e \equiv \{(e,\dot{e}) : \bar{q} \in \Omega\}$ such that $(e,\dot{e}) = (0,0) \in \Omega_e$, is asymptotically stable if there exist matrices $P_1 = P_1^T$, $P_2 = P_2^T$, M_1, M_2, $N_{\mathbf{j}}^1$, $N_{\mathbf{j}}^2$, $N_{\mathbf{j}}^3$, and $N_{\mathbf{j}}^4$, $\mathbf{j} \in \mathbb{B}^p$, of adequate dimensions such that the LMIs

$$\begin{bmatrix} P_1 & P_2 \\ P_2 & P_2 \end{bmatrix} > 0, \quad (14)$$

$$\sum_{\mathbf{ij} \in \mathcal{P}(\mathbf{kl})} \mathrm{He}\left(\begin{bmatrix} M_1 + N_{\mathbf{j}}^1 \bar{R}_{\mathbf{i}}^1 + N_{\mathbf{j}}^2 \bar{R}_{\mathbf{i}}^2 & P_1 + M_2 + N_{\mathbf{j}}^2 \bar{R}_{\mathbf{i}}^3 \\ M_1 + N_{\mathbf{j}}^3 \bar{R}_{\mathbf{i}}^1 + N_{\mathbf{j}}^4 \bar{R}_{\mathbf{i}}^2 & P_2 + M_2 + N_{\mathbf{j}}^4 \bar{R}_{\mathbf{i}}^3 \end{bmatrix}\right) < 0, \quad (15)$$

hold $\forall \mathbf{kl} \in \mathbb{B}^{2p+}$ with $\mathbb{B}^{2p+}$ being the set of ordered $2p$-bit binary numbers such that every bit is less or equal than the following, $\mathcal{P}(\mathbf{kl})$ being the set of indexes such that $\mathbf{w_i}(\bar{q})\mathbf{w_j}(\bar{q}) = \mathbf{w_k}(\bar{q})\mathbf{w_l}(\bar{q})$, and $\mathrm{He}(E) = E + E^T$. The controller gains in (3) are thus determined by $K_p = -P_2^{-1}M_1$ and $K_v = -P_2^{-1}M_2$.

Proof: The closed-loop error system resulting from substitution of (3) in (9) yields

$$\begin{bmatrix} \dot{e}(t) \\ \ddot{e}(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ -K_p & -K_v \end{bmatrix} \begin{bmatrix} e(t) \\ \dot{e}(t) \end{bmatrix}. \quad (16)$$

On the other hand, LMI condition (14) guarantees that

$$V(e,\dot{e}) = \begin{bmatrix} e \\ \dot{e} \end{bmatrix}^T \begin{bmatrix} P_1 & P_2 \\ P_2 & P_2 \end{bmatrix} \begin{bmatrix} e \\ \dot{e} \end{bmatrix} > 0, \forall \begin{bmatrix} e \\ \dot{e} \end{bmatrix} \neq 0,$$

which means $V(e,\dot{e})$ is a valid Lyapunov function candidate for (16); in order to be a Lyapunov function, its time derivative, once (16) is substituted, should hold:

$$\begin{aligned} \dot{V}(e,\dot{e}) =& 2 \begin{bmatrix} e \\ \dot{e} \end{bmatrix}^T \begin{bmatrix} P_1 & P_2 \\ P_2 & P_2 \end{bmatrix} \begin{bmatrix} \dot{e} \\ \ddot{e} \end{bmatrix} \\ =& \begin{bmatrix} e \\ \dot{e} \end{bmatrix}^T \mathrm{He}\left(\begin{bmatrix} P_1 & P_2 \\ P_2 & P_2 \end{bmatrix} \begin{bmatrix} 0 & I \\ -K_p & -K_v \end{bmatrix} \right) \begin{bmatrix} e \\ \dot{e} \end{bmatrix} < 0, \end{aligned}$$

which, by Finsler's Lemma and taking into account the algebraic restriction (10), amounts to the existence of a matrix

$$N(\bar{q}) = \begin{bmatrix} N^1(\bar{q}) & N^2(\bar{q}) \\ N^3(\bar{q}) & N^4(\bar{q}) \end{bmatrix},$$

such that

$$\begin{aligned} &\mathrm{He}\left(\begin{bmatrix} P_1 & P_2 \\ P_2 & P_2 \end{bmatrix} \begin{bmatrix} 0 & I \\ -K_p & -K_v \end{bmatrix} \right) \\ &+ \mathrm{He}\left(\begin{bmatrix} N^1(\bar{q}) & N^2(\bar{q}) \\ N^3(\bar{q}) & N^4(\bar{q}) \end{bmatrix} \begin{bmatrix} \bar{R}^1(\bar{q}) & 0 \\ \bar{R}^2(\bar{q}) & \bar{R}^3(\bar{q}) \end{bmatrix} \right) < 0, \end{aligned} \quad (17)$$

but considering the equivalence of (10) and (13) as well as the following choice of matrices in $N(\bar{q})$ above

$$N^i(\bar{q}) = \sum_{\mathbf{j}\in\mathbb{B}^p} \mathbf{w}_\mathbf{j}(\bar{q}), \quad i \in \{1,2,3,4\},$$

we have that condition (17) can be further rewritten as

$$\begin{aligned} &\sum_{\mathbf{i}\in\mathbb{B}^p} \sum_{\mathbf{j}\in\mathbb{B}^p} \mathbf{w}_\mathbf{i}(\bar{q}) \mathbf{w}_\mathbf{j}(\bar{q}) \mathrm{He}\left(\begin{bmatrix} -P_2K_p & P_1 - P_2K_v \\ -P_2K_p & P_2 - P_2K_v \end{bmatrix} \right. \\ &\left. + \begin{bmatrix} N_\mathbf{j}^1 \bar{R}_\mathbf{i}^1 + N_\mathbf{j}^2 \bar{R}_\mathbf{i}^2 & N_\mathbf{j}^2 \bar{R}_\mathbf{i}^3 \\ N_\mathbf{j}^3 \bar{R}_\mathbf{i}^1 + N_\mathbf{j}^4 \bar{R}_\mathbf{i}^2 & N_\mathbf{i}^4 \bar{R}_\mathbf{i}^3 \end{bmatrix} \right) < 0, \end{aligned} \quad (18)$$

where convex properties were employed to write the convex sums related to N^j and $\bar{R}^i$ at the outermost left side of the inequality.

LMIs (15) guarantee condition (18) as it can be checked by performing substitutions $M_1 = -P_2K_p$ and $M_2 = -P_2K_v$ in the latter, associating terms sharing the same factorization $\mathbf{w}_\mathbf{i}(\bar{q})\mathbf{w}_\mathbf{j}(\bar{q})$, and asking each association (indexed by $\mathcal{P}(\mathbf{kl})$) to be negative-definite, while dropping the products of convex functions $\mathbf{w}_\mathbf{i}(\bar{q})\mathbf{w}_\mathbf{j}(\bar{q})$ because they are positive. □

Remark 1: The solution space of Theorem 1 includes ordinary computed-torque as a particular case: indeed, if all matrices $N_\mathbf{j}^i$, $i \in \{1,2,3,4\}$, $\mathbf{j} \in \mathbb{B}^p$, are made zero, and K_p, K_v are made diagonal, the design conditions in [1] are recovered.

Remark 2: The LMI framework allows easily including decay rate, input constraints, and other performance specifications into the design by simply adding the corresponding LMI expressions. The interested reader is referred to [24] for details.

IV. Example

Consider a 2-DOF planar parallel manipulator acting on a horizontal plane –as schematically shown in Fig 1– with two active joints (θ_{11} and θ_{21}) and two passive ones (θ_{12} and θ_{22}), the end effector being located at point E. As explained before, the dynamical model requires virtually cutting at point E to model two open kinematic chains which, naturally, will have a practically identical model, namely [32]:

$$\begin{aligned} &\begin{bmatrix} \alpha_1 & 0 & \beta_1 C_1 & 0 \\ 0 & \alpha_2 & 0 & \beta_2 C_2 \\ \beta_1 C_1 & 0 & \delta_1 & 0 \\ 0 & \beta_2 C_2 & 0 & \delta_2 \end{bmatrix} \ddot{q}(t) \\ &+ \begin{bmatrix} 0 & 0 & \beta_1 S_1 \dot{\theta}_{12} & 0 \\ 0 & 0 & 0 & \beta_2 S_2 \dot{\theta}_{22} \\ -\beta_1 S_1 \dot{\theta}_{11} & 0 & 0 & 0 \\ 0 & -\beta_2 S_2 \dot{\theta}_{21} & 0 & 0 \end{bmatrix} \dot{q}(t) = \begin{bmatrix} \tau_1 \\ \tau_2 \\ 0 \\ 0 \end{bmatrix}, \end{aligned} \quad (19)$$

where $q = [\theta_{11}\, \theta_{21}\, \theta_{12}\, \theta_{22}]^T$ is the joint vector, $\alpha_i = m_{i1} r_{i1}^2 + Iz_{i1} + m_{i2} l_{i1}^2$, $\beta_i = m_{i2} l_{i1} r_{i2}$, and $\delta_i = m_{i2} r_{i2}^2 + Iz_{i2}$, with $l_{11} = l_{21} = 0.102$ and $l_{12} = l_{22} = 0.18$ being the link lengths, $m_{11} = m_{21} = 0.8$ and $m_{12} = m_{22} = 1.2$ being the links masses, $Iz_{11} = Iz_{21} = 0.0027$ and $Iz_{12} = Iz_{22} = 0.0013$ being the inertia tensors, and $r_{11} = r_{21} = 0.05$ and $r_{12} = r_{22} = 0.09$ being the distances from the joints to the centre of mass of the corresponding link; $C_i = \cos(\theta_{i1} - \theta_{i2})$ and $S_i = \sin(\theta_{i1} - \theta_{i2})$, $i = 1, 2$.

The following algebraic restrictions "close" the kinematic chains above by making their coordinates coincide at E:

$$\begin{aligned} l_{11}\cos\theta_{11} + l_{12}\cos\theta_{12} - l_{21}\cos\theta_{21} - l_{22}\cos\theta_{22} = 0, \\ l_{11}\sin\theta_{11} + l_{12}\sin\theta_{12} - l_{21}\sin\theta_{21} - l_{22}\sin\theta_{22} = 0. \end{aligned} \quad (20)$$

In order to write the error system (9) as in the second case in section III, we solve for $\ddot{\theta}_{12}$ and $\ddot{\theta}_{22}$ in the third and fourth equations in (19); then we substitute them in the first and second equations. In the latter 2, we replace any occurrence of $\dot{\theta}_{12}$

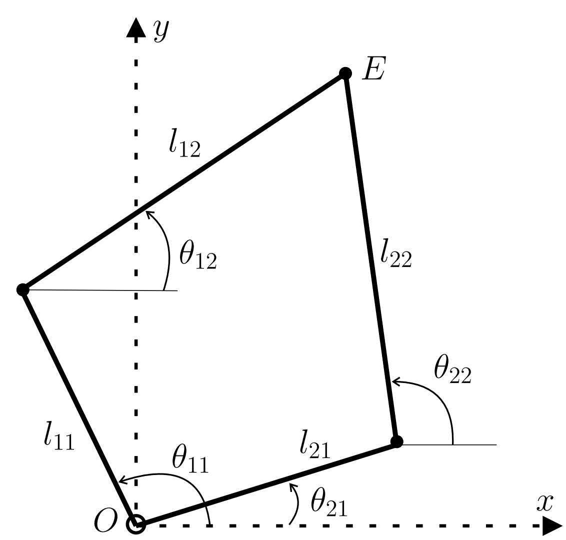

Fig. 1. Diagram of the 2-DOF parallel robot.

and $\dot{\theta}_{22}$ by their equivalence obtained from the time derivative of (20) and any occurrence of θ_{12} and θ_{22} by their equivalence from (20). As a result, the system (19) will adopt the form (7) with $\tilde{q} = [\theta_{11}\,\theta_{21}]^T$ and $\tilde{\tau}(t) = [\tau_1\,\tau_2]^T$; expressions in $\tilde{M}(\tilde{q})$ and $\tilde{V}(\tilde{q}, \dot{\tilde{q}})$ (the other vectors are 0) are too large to fit here; yet, as a token of illustration consider the entry $(1,1)$: $\tilde{M}_{11} = Iz_{11} + l_{11}^2 m_{21} + m_{11} r_{11}^2 - (l_{11}^2 m_{21}^2 r_{21}^2 \cos(2\arctan(c_2/c_3) - \theta_{11})^2)/(m_{21} r_{21}^2 + Iz_{21})$, with $c_1 = ((l_{11}^2 \cos(\theta_{11} - \theta_{21}) - l_{11}^2 + 2l_{21}^2)/(\cos(\theta_{11}/2)^2 \cos(\theta_{21}/2)^2))^{1/2}$, $c_2 = \sqrt{2}\cos(\theta_{11}/2 - \theta_{21}/2)c_1 - 4l_{21}\cos(\theta_{11}/2 + \theta_{21}/2) + \sqrt{2}\cos(\theta_{11}/2 + \theta_{21}/2)c_1$, and $c_3 = 4l_{11}\sin(\theta_{11}/2 - \theta_{21}/2) + 4l_{21}\sin(\theta_{11}/2 + \theta_{21}/2)$.

The algebraic restrictions imply that $R(\tilde{q}_d)\tilde{q}_d - R(\tilde{q})\tilde{q} = 0$ can be written as $\bar{R}^1(\tilde{q}, \tilde{q}_d)\tilde{e}$ by means of the factorization in [31], where $\tilde{q}_d = [\tilde{\theta}_{11}\,\tilde{\theta}_{21}]^T$, $\tilde{e} = \tilde{q}_d - \tilde{q}$, and

$$\bar{R}^1 = \begin{bmatrix} -0.5 l_{11}(\theta_{11} + \theta_{d11}) & 0.5 l_{21}(\theta_{21} + \theta_{d21}) \\ \frac{-l_{11}}{6}(\theta_{11}^2 + \theta_{11}\theta_{d11} + \theta_{d11}^2 - 6) & \frac{l_{21}}{6}(\theta_{21}^2 + \theta_{21}\theta_{d21} + \theta_{d21}^2 - 6) \end{bmatrix}$$

A convex representation of $\bar{R}^1(\tilde{q}, \tilde{q}_d)$ can be easily found as suggested in (13); the bounds $\tilde{q}_i, \tilde{q}_{di} \in [0\ \pi]$, $i = 1, 2$ were taken into account for that matter, yielding 2^8 different matrices $\bar{R}^1_{\mathbf{i}}$, $\mathbf{i} \in \mathbb{B}^4$; some of these matrices are:

$$\bar{R}^1_{0100} = \begin{bmatrix} 0 & 0 \\ 0.102 & 0.102 \end{bmatrix} \quad \bar{R}^1_{1001} = \begin{bmatrix} -0.32 & -0.32 \\ -0.57 & -0.57 \end{bmatrix}$$

If no algebraic restrictions are considered, the gains K_p and K_v for the reduced Lagrange-Euler model yield a diagonal ordinary form $K_p = \text{diag}\{1.5, 1.5\}$, $K_v = \text{diag}\{2.5, 2.5\}$, which has to be used in $u(t)$ for a computer-torque control law which only affects the actuated entries as in (8) with $\tilde{F}(\tilde{q}, \dot{\tilde{q}}) = \tilde{G}(\tilde{q}) = 0$. This would be a solution in the ordinary computed-torque framework, which does not exploit all the entries in the matrix gains.

We not turn our attention to considering algebraic restrictions into the design of the gains K_p and K_v; LMIs (14)-(15) are feasible as conditions in theorem 1 are fulfilled with the following solution:

$$P_1 = \begin{bmatrix} 256.4 & -0.007 \\ -0.004 & 256.4 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 47.48 & 0 \\ 0 & 47.48 \end{bmatrix},$$
$$M_1 = \begin{bmatrix} -84.48 & 168.96 \\ -168.96 & -84.48 \end{bmatrix}, \quad M_2 = \begin{bmatrix} -131.9 & 168.9 \\ -168.9 & -131.9 \end{bmatrix},$$
$$K_p = \begin{bmatrix} 1.78 & -3.56 \\ 3.56 & 1.78 \end{bmatrix}, \quad K_v = \begin{bmatrix} 2.78 & -3.56 \\ 3.56 & 2.78 \end{bmatrix}.$$

The desired trajectory $\tilde{q}_d$ is such that the end-effector located at point E follows the desired trajectory in (x, y) coordinates $x_d = 0.01\cos(t)$ and $y_d = 0.19 + 0.01\sin(t)$, describing a circle with center at $(0, 0.19)$ and radius 0.01. Fig. 2 shows the time evolution of θ_{11} and θ_{21} which asymptotically track θ_{d11} and θ_{d21}, respectively, from the initial conditions $q(0) = [2.3\,1.1\,0.7\,1.9]^T$. Fig. 3 shows the control signals $\tau_1(t)$ and $\tau_2(t)$ as well as the algebraic restrictions (20) which must be zero along the whole process in order to meet the DAE conditions. As expected, a full exploitation of the entries in the matrix gains is achieved; tracking is adequately performed[1].

[1]A computer animation can be found in: https://drive.google.com/file/d/1XXbYMNNoLh6ArS0pxl8lTUNNlkKBLIcs/view?usp=sharing

V. Conclusions

A novel computed-torque technique for a class of parallel robots that can be modelled as differential algebraic equations has been presented. It has been shown that the proposal enriches the set of gains that can be employed for the outer-loop feedback by including the algebraic restrictions in the design conditions. Linear matrix inequalities were obtained by combining the Finsler's Lemma and the direct Lyapunov method, which means a solution can be reached in polynomial time if it exists. The novel technique has been illustrated on a 2-degree-of-freedom planar parallel manipulator which has been made asymptotically follow a desired trajectory in its workspace. Future work will consider extending the proposal to any number of passive or active joints.

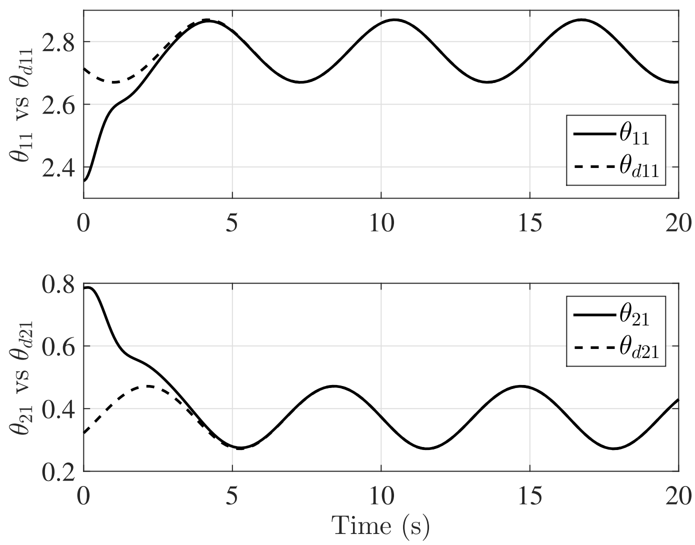

Fig. 2. Trajectory tracking of $\tilde{q}$ to $\tilde{q}_q$.

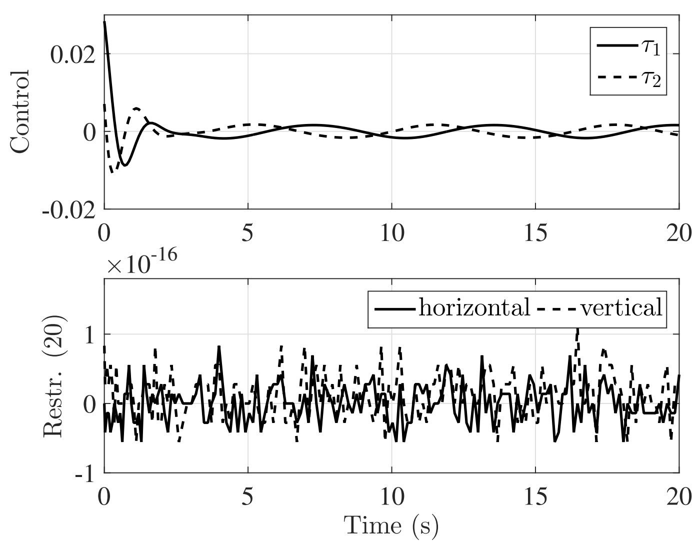

Fig. 3. Control signals and the algebraic restrictions (20).

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