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Backstepping Design for the Predefined-Time Stabilization of Second-Order Systems

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***Abstract*—The backstepping design of a controller which stabilizes a class of second-order systems in predefined-time is studied in this paper. The origin of a dynamical system is said to be predefined-time stable if it is fixed-time stable and an upper bound of the settling-time function can be *arbitrarily* chosen a priori through an appropriate selection of the system parameters. The proposed backstepping construction is based on recently proposed Lyapunov-like sufficient conditions for predefined-time stability. Different from other approaches, the proposed backstepping design allows the simultaneous construction of a Lyapunov function which meets the conditions for guaranteeing predefined-time stability. A simulation example is presented to show the behavior of a developed controller, and to show its advantages against similar schemes.**

***Index Terms*—Backstepping, Fixed-time stability, Lyapunov design, Predefined-time stability, Second-order systems.**

I. Introduction

The notions of Lyapunov and asymptotic stability have been widely studied for more than 100 years [1]. So it is not for less that several engineering problems have found their solutions, and some nature phenomena have been modeled and explained using these concepts.

On the other hand, several sophisticated industrial applications as batch processes control and monitoring, faults isolation, among others, demand the satisfaction of time-response constraints to satisfy safety, regulatory or quality standards. Nature itself presents us with some phenomena, like dry friction, which cannot be adequately modeled using the smooth fields associated with the mentioned types of stability. In this sense, the notion of *finite-time stability* has attracted a lot of attention during the last 50 years [2]–[4].

However, systems exhibiting the finite-time stability property generally have a settling time which, although finite, is unbounded as a function of the initial condition. A desired characteristic is to eliminate this boundlessness condition, for instance in estimation [5], [6], control [7], [8] or real-time optimization problems [9]. With this in mind, a stronger form of stability, called *fixed-time stability* has been studied in [10]–[13]. For systems showing the fixed-time stability property, the settling-time function is bounded.

Although conceptually the notion of fixed-time stability has some advantages over the concept of finite-time stability, it cannot be guaranteed in general that the convergence time can be arbitrarily selected through the system tunable parameters. Therefore, to surmount this difficulty, a new concept, called *predefined-time stability*, has been studied in [14], [15]. For systems presenting the predefined-time stability property, an upper bound of the settling-time function can be arbitrarily chosen through a suitable selection of the parameters of the system.

In this sense, controllers which induce the predefined-time stability on first-order systems [14]–[16], second-order systems [17]–[19] and non-holonomic systems [20] have been studied. In particular, for second-order systems, in [18] a time-based switching scheme is proposed to avoid some singularity issues and in [19], the controller is constructed under the block control procedure. Nevertheless, none of these approaches allows the construction of a Lyapunov function of predefined-time stability for systems whose order is greater than one.

Taking this into account, this paper presents a modified backstepping controller design for the predefined-time stabilization of a class of second-order systems, allowing, in turn, the simultaneous construction of a Lyapunov function of predefined-time stability for this class of systems. This controller also provides robustness against matched perturbations; namely, the closed-loop system is robust, even insensitive, in the presence of such perturbations. Given that predefined-time stability is a stronger form of finite-time (fixed-time) stability, it is not possible to induce it without using non-smooth control signals and, because of this reason, the standard backstepping procedure fails [21]. That is why a supplementary variable is defined, and the backstepping scheme is modified. The proposed construction relies on recently introduced Lyapunov-like sufficient conditions for predefined-time stability [16]. Simulation examples are presented to show the behavior of a developed controller, and to compare its performance against similar schemes [12].

II. Preliminaries

A. Notation

We use the following notation throughout the paper:

- $\mathbb{R}$ is the set of real numbers; $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$; $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$; $\bar{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$.
- For $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{x}^T$ denotes its transpose; $||\boldsymbol{x}|| = \sqrt{\boldsymbol{x}^T \boldsymbol{x}}$; $B_r(\boldsymbol{x}) = \{\boldsymbol{y} \in \mathbb{R}^n : ||\boldsymbol{y} - \boldsymbol{x}|| < r\}$.

- The function $x \to \lfloor x \rceil^h$ is defined as $\lfloor x \rceil = |x|^h \mathrm{sign}(x)$ for any $x \in \mathbb{R}$ if $h > 0$, and for any $x \in \mathbb{R} \setminus \{0\}$ if $h \le 0$.
- $\theta'(z) = \frac{d\theta}{dz}$ and $\theta''(z) = \frac{d^2\theta}{dz^2}$ denote the first and the second derivative, respectively, of the function $\theta : \mathbb{R} \to \mathbb{R}$.

B. Class $\mathcal{K}^1$ and class $\mathcal{W}$ functions

Inspired in the class-$\mathcal{K}$ functions in [22, Definition 1] and [23, Definition 4.2], the class-$\mathcal{K}^1$ functions are defined as follows:

Definition 1. A scalar continuous function $\kappa : \mathbb{R}_{\ge 0} \to [0, 1)$ is said to belong to class $\mathcal{K}^1$, denoted as $\kappa \in \mathcal{K}^1$, if it is strictly increasing, $\kappa(0) = 0$ and $\kappa(r) \to 1$ as $r \to \infty$.

It follows directly from the definition that $\mathcal{K}^1$ functions are bijective. In fact, $\kappa \in \mathcal{K}^1$ is injective because it is continuous and strictly increasing and its image is $\kappa(\mathbb{R}_{\ge 0}) = [0, 1)$, thus it is surjective. Thus, since every class $\mathcal{K}^1$ function is bijective, their inverses exist.

Definition 2. A scalar continuous function $\omega : \mathbb{R}_{\ge 0} \to [0, 1)$ is said to belong to class $\mathcal{W}$, denoted as $\omega \in \mathcal{W}$, if:

(i) $\omega \in \mathcal{K}^1$ is twice differentiable in $\mathbb{R}_+$,
(ii) $\omega'(r) > 0$ for $r > 0$ and $\omega'(0) \in \bar{\mathbb{R}}_+$, and
(iii) $\omega''(r) < 0$ for all $r > 0$.

Example 1. Let $0 < q < 1$. Some examples of class $\mathcal{W}$ functions are:

(i) $\omega(r) = 1 - \exp(-r^q)$;
(ii) $\omega(r) = \frac{2}{\pi} \arctan(r^q)$; and
(iii) $\omega(r) = \frac{r^q}{r^q + \alpha}$, with $\alpha > 0$.

C. On predefined-time stability

Consider the following autonomous system:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}; \boldsymbol{\rho}), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the system state, the vector $\boldsymbol{\rho} \in \mathbb{R}^l$ stands for the *tunable* parameters of (1). The function $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ may be discontinuous, and such that the solutions of (1) exist and are unique in the sense of Filippov (see [24] and [25, Proposition 5]). Thus, $\boldsymbol{\Phi}(t, \boldsymbol{x}_0)$ denotes the solution of (1) starting from $\boldsymbol{x}_0 \in \mathbb{R}^n$ at $t = 0$. Moreover, the origin $\boldsymbol{x} = \boldsymbol{0}$ is the unique equilibrium point of (1).

Remark 1. Usually, the parameters of a given dynamical system are not tunable, but they have fixed numerical values instead. On the other hand, in control or observation problems, for example, we consider systems of the form

$$\dot{\boldsymbol{x}} = \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{\phi}(\boldsymbol{x}; \boldsymbol{\rho})), \tag{2}$$

where $\boldsymbol{\rho}$ are tunable parameters of which function $\boldsymbol{\phi}$ depends. The functions $\boldsymbol{\phi} : \mathbb{R}^n \to \mathbb{R}^m$ and $\boldsymbol{g} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ may represent:

- in control design: $\boldsymbol{\phi}$ stands for the feedback control input to be designed, and $\boldsymbol{g}$ represents the closed-loop dynamics of the tracking error $\boldsymbol{x}$;
- in observer design: $\boldsymbol{\phi}$ represents the feedback error injection to be designed, whereas $\boldsymbol{g}$ represents the dynamics of the estimation error $\boldsymbol{x}$.

In any case, we may study the properties of (2) as if we were studying the properties of the autonomous system (1), defining $\boldsymbol{f}(\boldsymbol{x}; \boldsymbol{\rho}) := \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{\phi}(\boldsymbol{x}; \boldsymbol{\rho}))$.

Definition 3 ([26])**.** The origin of (1) is said to be

- **Lyapunov stable** if for any $\boldsymbol{x}_0 \in \mathbb{R}^n$, the solution $\boldsymbol{\Phi}(t, \boldsymbol{x}_0)$ is defined for all $t \ge 0$, and for any $\epsilon > 0$, there is $\delta > 0$ such that for any $\boldsymbol{x}_0 \in \mathbb{R}^n$, if $\boldsymbol{x}_0 \in B_\delta(\boldsymbol{0})$ then $\boldsymbol{\Phi}(t, \boldsymbol{x}_0) \in B_\epsilon(\boldsymbol{0})$ for all $t \ge 0$;
- **asymptotically stable** if it is Lyapunov stable and $\boldsymbol{\Phi}(t, \boldsymbol{x}_0) \to 0$ as $t \to \infty$, for any $\boldsymbol{x}_0 \in \mathbb{R}^n$;
- **finite-time stable** if it is Lyapunov stable and for any $x_0 \in \mathbb{R}^n$ there exists $0 \le T < \infty$ such that $\boldsymbol{\Phi}(t, x_0) = \boldsymbol{0}$ for all $t \ge T$. The function $T(\boldsymbol{x}_0) = \inf \{T \ge 0 : \boldsymbol{\Phi}(t, \boldsymbol{x}_0) = \boldsymbol{0}, \forall t \ge T\}$ is called the **settling-time function** of (1);
- **fixed-time stable** if it is finite-time stable and the settling-time function of (1), $T(\boldsymbol{x}_0)$, is bounded on $\mathbb{R}^n$, i.e. there exists $T_{\max}$ such that $\sup_{\boldsymbol{x}_0 \in \mathbb{R}^n} T(\boldsymbol{x}_0) \le T_{\max} < \infty$

Example 2. Consider the system (1) having the form

$$\dot{x} = -\frac{1}{\rho_1} \lfloor x \rceil^{\rho_2} - \rho_1 \lfloor x \rceil^{2-\rho_2}, \tag{3}$$

where $x \in \mathbb{R}$ is the state of the system, $\boldsymbol{\rho} = [\rho_1,\, \rho_2]^T \in \mathbb{R}^2$ is the vector of tunable parameters of (3), which comply to $\rho_1 > 0$ and $0 < \rho_2 < 1$. Using [12, Lemma 1], one can easily show that the origin of (3) is fixed-time stable. Moreover, from [27, Theorem 1], the settling-time function of (3) satisfies

$$\sup_{x_0 \in \mathbb{R}} T(x_0) = \frac{\mathcal{B}\left(1/2, 1/2\right)}{2\rho_1^{-1/2} \rho_1^{1/2} (1 - \rho_2)} = \frac{\pi}{2(1 - \rho_2)} > \frac{\pi}{2}.$$

This example shows that the convergence time for system (3), whose origin is fixed-time stable, cannot be reduced arbitrarily no matter how the parameters $\boldsymbol{\rho}$ are tuned. The case when the convergence time can be arbitrarily assigned through an appropriate tuning of the system parameters $\boldsymbol{\rho}$ corresponds to the notion of predefined-time stability, which is defined as follows:

Definition 4. The origin of (1) is said to be **predefined-time stable** if it is fixed-time stable and for any $T_c \in \mathbb{R}_+$, there exists some $\boldsymbol{\rho} \in \mathbb{R}^l$ such that the settling-time function of (1) satisfies

$$\sup_{\boldsymbol{x}_0 \in \mathbb{R}^n} T(\boldsymbol{x}_0) \le T_c.$$

Example 3. Let the system (1) of the form given in [10], [12]

$$\begin{aligned} \dot{x} &= -\left\lfloor \rho_1 \lfloor x \rceil^{\rho_3} + \rho_2 \lfloor x \rceil^{\rho_4} \right\rceil^{\rho_5} \\ &= -\left(\rho_1 |x|^{\rho_3} + \rho_2 |x|^{\rho_4}\right)^{\rho_5} \mathrm{sign}(x), \end{aligned} \tag{4}$$

where $x \in \mathbb{R}$ is the state of the system, $\boldsymbol{\rho} = [\rho_1,\, \rho_2,\, \rho_3,\, \rho_4,\, \rho_5]^T \in \mathbb{R}^5$ is the vector of tunable parameters of (4), which comply to $\rho_1, \rho_2, \rho_5 > 0$ and $0 < \rho_5\rho_3 < 1 < \rho_5\rho_4$. The origin of (4) is fixed-time stable, by [12, Lemma 1]. Moreover, given $T_c \in \mathbb{R}_+$, there exist $\rho_1 = \rho_2 = \frac{\Gamma(1/4)^4}{4\pi T_c^2}$, $\rho_3 = 1$, $\rho_4 = 3$ and $\rho_5 = \frac{1}{2}$, such that the settling-time function of (4) satisfies (see [27, Theorem 1]) $\sup_{x_0 \in \mathbb{R}} T(x_0) =$

$\frac{\Gamma(1/4)^2}{\left(\frac{\Gamma(1/4)^4}{4\pi T_c^2}\right)^{1/2}\Gamma(1/2)(3-1)} = T_c$. Thus, the origin of system (4) is predefined-time stable.

The following Lyapunov-like theorem provides sufficient conditions for a system to present the predefined-time stability property.

Theorem 1. *Let $\kappa \in \mathcal{K}^1$ be differentiable in $\mathbb{R} \setminus \{0\}$, and $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a continuous, radially unbounded and positive definite function. If for any $T_c \in \mathbb{R}_+$, there exists some $\boldsymbol{\rho} \in \mathbb{R}^l$, such that the time-derivative of V along the trajectories of* (1) *satisfies*

$$\dot{V}(\boldsymbol{x}) \leq -\frac{1}{T_c \kappa'(V(\boldsymbol{x}))}, \quad \text{for } \boldsymbol{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}, \tag{5}$$

then the origin of (1) *is predefined-time stable. Moreover, if* (5) *is an equality, then* $\sup_{\boldsymbol{x}_0 \in \mathbb{R}^n} T(\boldsymbol{x}_0) = T_c$.

Proof. Let $T_c \in \mathbb{R}_+$. Then, by hypothesis, there exists some $\boldsymbol{\rho} \in \mathbb{R}^l$ such that (5) holds. Given that function V is a continuous, positive definite and radially unbounded function, the origin ofsystem (1) is asymptotically stable [23].

Now, consider the function $W : \mathbb{R}^n \to [0, 1)$, defined by $W(\boldsymbol{x}) = \kappa(V(\boldsymbol{x}))$. Hence, using (5), the time-derivative of $W(\boldsymbol{x})$ along the trajectories of (1) satisfies $\dot{W}(\boldsymbol{x}) \leq -\frac{1}{T_c}$ for $\boldsymbol{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. Thus, let $\boldsymbol{\Phi}(t, \boldsymbol{x}_0)$ be a solution of (1) and let $w(t) \geq 0$ be a function that satisfies $\dot{w} = -\frac{1}{T_c}$, and $W(\boldsymbol{x}_0) \leq w(0)$. Hence,

$$w(t) = \begin{cases} w(0) - \frac{t}{T_c} & \text{if} \quad 0 \leq t \leq T_c w(0) \\ 0 & \text{if} \quad t > T_c w(0), \end{cases}$$

and $W(\boldsymbol{\Phi}(t, \boldsymbol{x}_0)) \leq w(t)$ (it is an equality only if (5) is an equality) by the comparison lemma [23]. Thus, $W(\boldsymbol{\Phi}(t, \boldsymbol{x}_0)) = 0$ for $t \geq T_c W(\boldsymbol{x}_0)$, implying that the trajectories of (1) reach the origin in finite time, and the settling-time function satisfies $\sup_{\boldsymbol{x}_0 \in \mathbb{R}^n} T(\boldsymbol{x}_0) \leq \sup_{\boldsymbol{x}_0 \in \mathbb{R}^n} T_c W(\boldsymbol{x}_0) = T_c$. Hence, by Definition 4, the origin of system (1) is predefined-time stable. Moreover, if (5) is an equality, then $\sup_{\boldsymbol{x}_0 \in \mathbb{R}^n} T(\boldsymbol{x}_0) = \sup_{\boldsymbol{x}_0 \in \mathbb{R}^n} T_c W(\boldsymbol{x}_0) = T_c$. □

III. Predefined-time backstepping controller for second-order systems

A. Problem statement

Consider the generic class of second-order dynamical systems

$$\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y}) + \boldsymbol{g}(\boldsymbol{y})(\bar{u} + d), \tag{6}$$

where $\boldsymbol{y} \in \mathbb{R}^2$, $\bar{u} \in \mathbb{R}$, $\boldsymbol{f}, \boldsymbol{g} : \mathbb{R}^2 \to \mathbb{R}^2$ are smooth vector fields, and $d \in \mathbb{R}$ is a matched term which stands for all external perturbations and uncertainties.

Under the assumption that $\operatorname{span}\{(\partial \boldsymbol{g}/\partial \boldsymbol{y})\boldsymbol{f}(\boldsymbol{y}) - (\partial \boldsymbol{f}/\partial \boldsymbol{y})\boldsymbol{g}(\boldsymbol{y}), \boldsymbol{g}(\boldsymbol{y})\}$ is a non-singular distribution whose dimension is equal to 2, system (6) can be transformed into [28]

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u + \Delta, \end{aligned} \tag{7}$$

where $[x_1 \quad x_2]^T = \boldsymbol{x} \in \mathbb{R}^2$ is the state, $u \in \mathbb{R}$ is the control input and $\Delta \in \mathbb{R}$ is an unknown but bounded perturbation term of the form $\sup_{t \in \mathbb{R}_{\geq 0}} |\Delta(t)| \leq \delta$, with $0 \leq \delta < \infty$ a known constant.

The objective is to design a feedback control input $u = u(\boldsymbol{x})$ such that (7) is predefined-time stable.

B. Controller construction

We will construct the controller in two steps following the backstepping procedure.

Step 1: consider the Lyapunov function candidate $V_1(x_1) = |x_1|$. Then, the time-derivative of V_1 along the trajectories of (7) is

$$\dot{V}_1(x_1) = \operatorname{sign}(x_1)x_2, \quad \text{for } x_1 \neq 0. \tag{8}$$

The variable x_2 is used as a virtual control input to stabilize x_1. To this end, let

$$s = x_2 - v_1(x_1), \tag{9}$$

where $v_1(x_1) = -\frac{\operatorname{sign}(x_1)}{\rho_1 \omega'(|x_1|)}$, with $\omega \in \mathcal{W}$, and $\rho_1 > 0$.

Hence, replacing (9), (8) becomes

$$\begin{aligned} \dot{V}_1(x_1) =& \operatorname{sign}(x_1)v_1(x_1) + \operatorname{sign}(x_1)s \\ =& -\frac{1}{\rho_1 \omega'(|x_1|)} + \operatorname{sign}(x_1)s, \quad \text{for } x_1 \neq 0. \end{aligned} \tag{10}$$

Note that if $s = 0$, then (10) takes the form of (5) (see Theorem 1).

Step 2: now, inspired in [12, Theorem 2], consider the function

$$s_d = x_2 + \left\lfloor \lfloor x_2 \rceil^2 - 2 \lfloor v_1(x_1) \rceil^2 \right\rceil^{1/2}. \tag{11}$$

The following proposition states that s_d vanishes in the same set and has the same signum as the function s in (9).

Proposition 1. *The variables s and s_d, in* (9) *and* (11)*, respectively, satisfy:*

(i) $s s_d \geq 0$;
(ii) $s s_d = 0$ *if and only if* $s = s_d = 0$.

Proof. Given that the function $\lfloor \cdot \rceil^h$ is increasing, we have that

$$\begin{aligned} s_d \geq 0 \Longleftrightarrow& x_2 \geq -\left\lfloor \lfloor x_2 \rceil^2 - 2 \lfloor v_1(x_1) \rceil^2 \right\rceil^{1/2} \\ \Longleftrightarrow& \lfloor x_2 \rceil^2 \geq -\lfloor x_2 \rceil^2 + 2 \lfloor v_1(x_1) \rceil^2 \\ \Longleftrightarrow& \lfloor x_2 \rceil^2 \geq \lfloor v_1(x_1) \rceil^2 \\ \Longleftrightarrow& x_2 \geq v_1(x_1) \\ \Longleftrightarrow& s = x_2 - v_1(x_1) \geq 0. \end{aligned}$$

Hence, proposition *(i)* is proved. Moreover, proposition *(ii)* follows from identical steps, but replacing the inequality symbol "$\geq$" by the equality symbol "$=$". □

Considering the above, let

$$V_2(x_1, s_d) = |x_1| + |s_d| = V_1(x_1) + |s_d| \tag{12}$$

be a Lyapunov function candidate. Its time-derivative along the trajectories of system (7) is

$$\begin{aligned}\dot{V}_2(x_1,s_d) &= \dot{V}_1(x_1) + \text{sign}(s_d)\dot{s}_d \\ &= -\frac{1}{\rho_1\omega'(|x_1|)} + \text{sign}(x_1)s \\ &\quad + \text{sign}(s_d)\Bigg[u+\Delta \\ &\quad + \frac{|x_2|\,(u+\Delta) - s\,|v_1(x_1)v_1'(x_1)x_2|}{\left|\lfloor x_2\rceil^2 - 2\lfloor v_1(x_1)\rceil^2\right|^{1/2}}\Bigg] \end{aligned} \tag{13}$$

for $x_1, s_d \neq 0$. Thus, the control input u is designed as

$$\begin{aligned}u &= v_2(s_d) - \rho_2\text{sign}(s_d) - s \\ &\quad + 2\,|v_1(x_1)|\,v_1'(x_1)\text{sign}(s_d), \end{aligned} \tag{14}$$

where $v_2(s_d) = -\frac{\text{sign}(s_d)}{\rho_1\omega'(|s_d|)}$, and $\rho_2 \geq \delta$.

Using the fact that $\text{sign}(s) = \text{sign}(s_d)$ (see Proposition 1), and replacing (14) in (13), it yields

$$\begin{aligned}\dot{V}_2(x_1,s_d) &= -\frac{1}{\rho_1\omega'(|x_1|)} - \frac{1}{\rho_1\omega'(|s_d|)} + 2\,|v_1(x_1)|\,v_1'(x_1) \\ &\quad + \text{sign}(x_1)s - (k+1)\,|s| - \rho_2 + \Delta\text{sign}(s_d) \\ &\quad - \frac{\frac{|x_2|}{\rho_1\omega'(|s_d|)} + (k+1)\,|x_2|\,|s|}{\left|\lfloor x_2\rceil^2 - 2\lfloor v_1(x_1)\rceil^2\right|^{1/2}} \\ &\quad + 2\,|v_1(x_1)|\,v_1'(x_1)\frac{|x_2| - x_2\text{sign}(s_d)}{\left|\lfloor x_2\rceil^2 - 2\lfloor v_1(x_1)\rceil^2\right|^{1/2}}.\end{aligned}$$

Note that:

(i) $v_1'(x_1) = \frac{\omega''(|x_1|)}{\rho_1(\omega'(|x_1|))^2} \leq 0$;

(ii) $\text{sign}(x_1)s \leq |\text{sign}(x_1)s| = |s|$ and $x_2\text{sign}(s_d) \leq |x_2\text{sign}(s_d)| = |x_2|$;

(iii) $\Delta\text{sign}(s_d) \leq |\Delta| \leq \delta \leq \rho_2$.

Taking *(i), (ii)* and *(iii)* into account, the time-derivative of V_2 along the trajectories of (7) satisfies

$$\dot{V}_2(x_1,s_d) \leq -\frac{1}{\rho_1\omega'(|x_1|)} - \frac{1}{\rho_1\omega'(|s_d|)}, \quad \text{for } x_1, s_d \neq 0. \tag{15}$$

From (15), the origin of system (7)-(14) is asymptotically stable. Moreover, for $V_2(x_1,s_d)) = |x_1| + |s_d|$, there are some choices of $\omega \in \mathcal{W}$ (see Proposition 2 in the Appendix) for which (15) satisfies

$$\dot{V}_2(x_1,s_d) \leq -\frac{1}{2\rho_1\omega'(V_2(x_1,s_d))} \tag{16}$$

for $x_1, s_d \neq 0$.

Remark 2. The function $V_2(x_1,s_d)$ is continuously differentiable everywhere except in the set $S = S_1 \cup S_2$, where $S_1 = \{(x_1,s_d) : x_1 = 0\}$ and $S_2 = \{(x_1,s_d) : s_d = 0\}$. Moreover, since $\dot{x}_1 = x_2$, the trajectories of (7) just cross the manifold S_1 and cannot remain on it, except at the origin $(x_1,x_2) = (0,0)$. On the other hand, a sliding mode might appear on the manifold S_2, due to the controller structure (14). If this is the case, note that system (7) would evolve according to the reduced order equation $\dot{x}_1 = x_2 = v_1(x_1)$, which is predefined-time stable (see (10)).

All the above analysis is summarized in the next theorem.

Theorem 2. *The origin of closed-loop system* (7)-(14) *is predefined-time stable.*

Proof. Let $T_c \in \mathbb{R}_+$. The time-derivative of the Lyapunov function (12) along trajectories the closed-loop system (7)-(14) satisfies (16). Hence, using Theorem 1, the origin of the closed-loop system (7)-(14) is predefined-time stable and the settling-time function complies to

$$\sup_{\boldsymbol{x}_0\in\mathbb{R}^2} T(\boldsymbol{x}_0) \leq T_c = 2\rho_1.$$

□

IV. Simulation Results

To illustrate the effectiveness of the proposed control scheme, consider the closed loop system (7)-(14), with the particular selection of $\omega(r) = \frac{2}{\pi}\arctan(r^q)$ for the controller (14).

In the simulation, the bounded perturbation term, which is unknown to the controller, has the form $\Delta(t) = \sin(10\pi t)$. Moreover, the controller (14) parameters are set to $q = 0.4$, $\rho_1 = 0.5$ to ensure a convergence time of $T_c = 1$ (see Theorem 2), and $\rho_2 = 2 \geq 1 = \delta \geq |\Delta(t)|$. It is worth to mention that all the simulations are conducted using the Euler integration method, with a fundamental step size of 1×10^{-5} s.

In the first place, the predefined-time convergence property is demonstrated simulating the closed-loop system (7)-(14) for the following initial conditions: $\boldsymbol{x}(0) = [x_0 \quad x_0]^T$, with $x_0 = 1, 10, 100, 1000$.

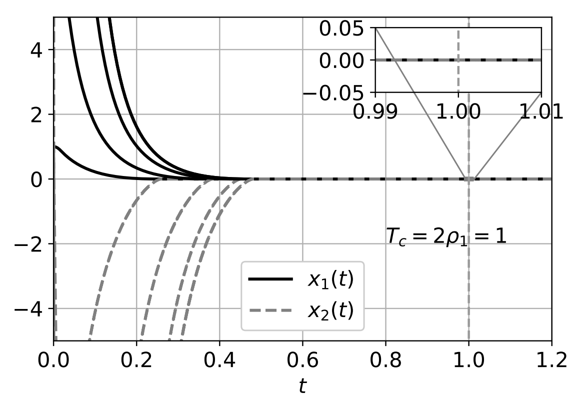

Fig. 1. State variables $x_1(t)$ and $x_2(t)$ over time, for several initial conditions of the form $\boldsymbol{x}(0) = [x_0 \quad x_0]^T$, with $x_0 = 1, 10, 100, 1000$.

Fig. 1 shows the evolution of the state variables over time, varying the initial conditions of the closed-loop system (7)-(14) as mentioned. One can appreciate the convergence of the variables to the origin before the predefined convergence time $T_c = 2\rho_1 = 1$.

In the second place, and for comparison purposes, we also include the simulation of the fixed-time (predefined-time, in fact) controller proposed in [12, Theorem 2], which has the following form

$$u = -\frac{\alpha_1 + 3\beta_1 x_1^2 + 2k}{2}\operatorname{sign}(\sigma) - \left\lfloor \alpha_2\sigma + \beta_2 \left\lfloor \sigma \right\rceil^3 \right\rceil^{1/2} \quad (17)$$

where $\sigma = x_2 + \left\lfloor \left\lfloor x_1 \right\rceil^2 + \alpha_1 x_1 + \beta_1 \left\lfloor x_1 \right\rceil^3 \right\rceil^{1/2}$.

In [12, Theorem 2], it is stated that the parameters should be calculated as $\frac{\alpha_1}{2} = \alpha_2 = \frac{\beta_1}{2} = \beta_2 = 64$ to ensure a convergence time of $T_c = 1$ s; moreover, the parameter $k = 2$.

Now, some advantages of the proposed scheme with respect to the controller (17) are shown simulating both schemes with the same the initial condition $\boldsymbol{x}(0) = [1000 \quad 0]^T$.

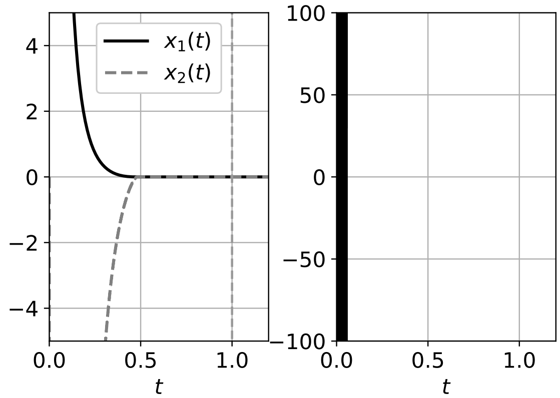

Fig. 2. State variables $x_1(t)$ and $x_2(t)$ over time (left) and control input signal (right) for the closed-loop system (7)-(14) with initial condition $\boldsymbol{x}(0) = [1000 \quad 0]^T$.

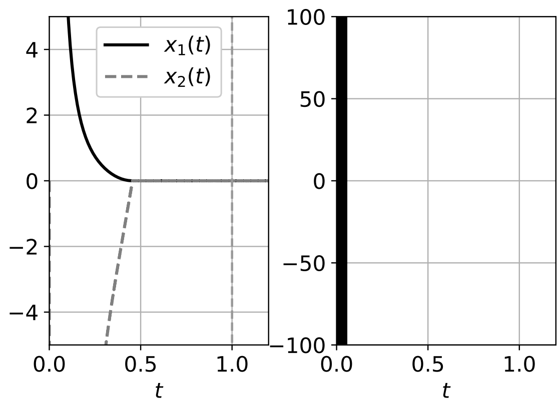

Fig. 3. State variables $x_1(t)$ and $x_2(t)$ over time (left) and control input signal (right) for the closed-loop system (7)-(17) with initial condition $\boldsymbol{x}(0) = [1000 \quad 0]^T$.

Figs. 2-3 show the evolution of the state variables over time and the control input signal for the controller schemes (14) and (17), respectively. Two things may be noted here:

(i) the convergence time estimation for the mentioned initial condition is the same for both schemes ($T_c = 1$ s); however, the actual convergence time for the proposed scheme is 0.48 s, whereas for the scheme proposed in [12, Theorem 2] it is 0.45 s. This is, the convergence time estimation with the proposed scheme is better.

(ii) The amplitude of the high frequency oscillations for the proposed controller is $\rho_2 = 2$ after the origin is reached, whereas for the controller proposed in [12, Theorem 2] it is $\frac{\alpha_1 + 2k}{2} = 66$. This is, the control effort (measured with the amplitude of the oscillations) needed by the proposed controller after reaching the origin represents only 3.03% of the control effort needed by the controller proposed in [12, Theorem 2].

V. Conclusion

This paper was dedicated to the design of a controller which stabilizes a class of second-order systems in predefined-time, using a modified backstepping construction procedure. Given that predefined-time stability is a stronger form of finite-time (fixed-time) stability, the standard backstepping procedure fails because of the non-smooth control signals. That is why inspired in [12, Theorem 2] a new variable was defined, and the backstepping procedure was modified.

Different from other approaches like [18], [19], the proposed controller synthesis allowed, for the first time, the construction of a Lyapunov function of predefined-time stability for second-order systems.

Finally, a simulation example validated the theoretical results showing the behavior of the proposed control scheme and its advantages against the controller proposed in [12, Theorem 2].

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Appendix
Some Important Inequalities

In this appendix, some important inequalities are reviewed.

Lemma 1 (L1). *Let $x_1, x_2 \in \mathbb{R}$ and $f, g : \mathbb{R} \to \mathbb{R}$ be two similarly ordered functions (i.e., f and g are both non-decreasing, or f and g are both non-increasing). Then, $f(x_1)g(x_1) + f(x_2)g(x_2) \geq \frac{1}{2}\left(f(x_1) + f(x_2)\right)\left(g(x_1) + g(x_2)\right)$.*

Proof. Since f and g are similarly ordered, the result follows from he Chebyshev's Inequality [29, Theorem 43]. □

Lemma 2 (L2). *Let $0 \leq q \leq 1$ and $x_1, x_2 \in \mathbb{R}_{\geq 0}$. Then,*

$$x_1^q + x_2^q \geq (x_1 + x_2)^q.$$

Proof. Consider the function $\epsilon : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, defined by $\epsilon(x, y) = x^q + y^q - (x + y)^q$. It is to be proved that $\epsilon(x_1, x_2) \geq 0$. First of all, note that $\epsilon(x, 0) = \epsilon(0, y) = 0$ for all $x, y \in \mathbb{R}_+ \cup \{0\}$. Furthermore, since $-1 \leq q - 1 \leq 0$, the partial derivatives $\frac{\partial \epsilon(x,y)}{\partial x} = q\left[x^{q-1} - (x + y)^{q-1}\right] \geq 0$ and $\frac{\partial \epsilon(x,y)}{\partial y} = q\left[y^{q-1} - (x + y)^{q-1}\right] \geq 0$ for $x, y \in \mathbb{R}_+$. Hence, $\epsilon(x_1, x_2) \geq 0$ and the proof is concluded. □

Now, we are ready to prove a subadditivity-like property for the following functions having at hand the previous inequalities.

Proposition 2. *Let $0 < q < \frac{1}{2}$. The class $\mathcal{W}$ functions:*
(i) $\omega(r) = \frac{2}{\pi}\arctan(r^q)$, *and*
(ii) $\omega(r) = \frac{r^q}{r^q+1}$,
satisfy $\frac{1}{\omega'(x)} + \frac{1}{\omega'(y)} \geq \frac{1}{2}\frac{1}{\omega'(x+y)}$ *for* $x, y \in \mathbb{R}_{\geq 0}$.

Proof. *(i)* Note that $\frac{1}{\omega'(r)} = \frac{\pi}{2q}(1 + r^{2q})r^{1-q}$. Hence, using Lemma 1 (L1) and Lemma 2 (L2), we have

$$\begin{aligned}\frac{1}{\omega'(x)} + \frac{1}{\omega'(y)} &= \frac{\pi}{2q}((1 + x^{2q})x^{1-q} + (1 + y^{2q})y^{1-q}) \\ &\geq \frac{1}{2}\frac{\pi}{2q}((2 + x^{2q} + y^{2q})(x^{1-q} + y^{1-q}))\ \text{(L1)} \\ &\geq \frac{1}{2}\frac{\pi}{2q}((2 + (x + y)^{2q})(x + y)^{1-q}\ \text{(L2)} \\ &\geq \frac{1}{2}\frac{\pi}{2q}((1 + (x + y)^{2q})(x + y)^{1-q}\ (2 \geq 1) \\ &= \frac{1}{2}\frac{1}{\omega'(x + y)}.\end{aligned}$$

(ii) Note that $\frac{1}{\omega'(r)} = \frac{1}{q}(1+r^q)^2 r^{1-q}$. Hence, using Lemma 1 (L1) and Lemma 2 (L2), we have

$$\begin{aligned}\frac{1}{\omega'(x)} + \frac{1}{\omega'(y)} &= \frac{1}{q}((1 + x^q)^2 x^{1-q} + (1 + y^q)^2 y^{1-q}) \\ &\geq \frac{1}{2}\frac{1}{q}((1 + x^q)^2 + (1 + y^q)^2)(x^{1-q} + y^{1-q}) \\ &\quad \text{(L1)} \\ &\geq \frac{1}{2}\frac{1}{q}((2 + 2(x + y)^q + (x + y)^{2q})(x + y)^{1-q} \\ &\quad \text{(L2)} \\ &= \frac{1}{2}\frac{1}{q}(1 + (x + y)^q)^2(x + y)^{1-q} \\ &\quad (2 \geq 1) \\ &= \frac{1}{2}\frac{1}{\omega'(x + y)}.\end{aligned}$$

□