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Rotary-Wing Aircraft Model for Control

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***Abstract*—A rotary wing aircraft is nonlinear system that has a complex interaction between the blades movement and the air flow; e.g. in translation, the advancing blade furnishes more lift than the retreating blade. The tilting of the disk and the feathering of the blades, used for control make the model not affine in the control and are part of the state. In this work we present a model that takes into account the interaction between the air flow and the blades, integrating the disk tilt and the blade pitch into the state. The control action in this case is the change in the disk tilt, the collective pitch and the disk speed of the main rotor and the change of the rotor speed and the pitch for the tail rotor. The rotary wing aircraft is modeled using Spatial Operator Algebra as a set of three rigid bodies, the airframe, the main rotor and the tail rotor whose individual dynamics are joined using Newton's Second law.**

I. Introduction

Numerous vehicles can be seen as a body subjected to external forces: aircraft, balloons, boats, helicopters, ships, submarines, etc. These vehicles are usually controlled by a person or group of persons. Our aim is to develop autonomous vehicles, so we need a model in order to design a control.

In this paper, we present the development of a helicopter model, a rotary wing aircraft. Since the main characteristic of the vehicle are its rotors, we model them as linked to the helicopter body with the corresponding degrees of freedom. This gives us a model that is not affine in the control. Therefore, the model must be transformed into an affine form.

The Newton-Euler approach usually used to model the dynamics[1], in this work the model is extended, considering the helicopter as an articulated multibody [2] [3], to include the gyroscopic effects and the asymmetric forces of the rotor. The aerodynamic forces on the rotor are approximated using the blade element theory [4] extended to tridimensional flow, considering the rotors as rigid bodies in this paper.

II. Dynamic Model

A. Spatial Operator Algebra Basics

The use of screws [5] to express generalized speeds and forces has become ubiquitous in robotics, extensions are spatial vector algebra [6] and spatial operator algebra [3]. The spatial speed (twist in screw theory) can defined by the projection of the angular and the linear speeds on an inertial frame or a body fixed frame with its origin at a point k of the body

$$V(k) = {}^{\mathbb{I}}\{\omega_{\mathtt{x}}, \omega_{\mathtt{y}}, \omega_{\mathtt{z}}, \mathtt{u}_{\mathtt{x}}, \mathtt{u}_{\mathtt{y}}, \mathtt{u}_{\mathtt{z}}\}^{\mathtt{T}} = {}^{\mathbb{B}}\{\mathtt{p}, \mathtt{q}, \mathtt{r}, \mathtt{u}, \mathtt{v}, \mathtt{w}\}^{\mathtt{T}} \quad (1)$$

and the spatial force (wrench in screw theory) by the torques and forces acting on point k

$$F(k) = \{M_x, M_y, M_z, F_x, F_y, F_z\}^T \quad (2)$$

The operator Φ is used to calculate the spatial speed and force acting on different points of a rigid body

$$V(r) = \Phi^*(k,r)V(k) \qquad F(r) = \Phi(r,k)F(k) \quad (3)$$

where

$$\Phi(r,k) = \begin{bmatrix} 1 & 0 & 0 & 0 & -(z_k - z_r) & y_k - y_r \\ 0 & 1 & 0 & z_k - z_r & 0 & -(x_k - x_r) \\ 0 & 0 & 1 & -(y_k - y_r) & x_k - x_r & \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (4)$$

is the *rigid body transformation matrix*. This expressions allow to calculate the spatial speeds and the spatial forces in a systematic way.

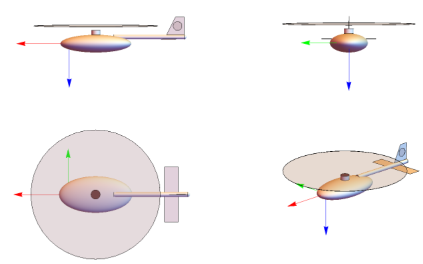

Fig. 1. Model used for the rotary-wing aircraft considered in the paper. The standard frame of reference for aircraft is used: positive directions of the (x, y, z) axes are represented as $\{red, green, blue\}$ arrows, with the origin at the reference point k.

B. Body Dynamics

The helicopter body dynamics is described by the equation of a rigid body

$$\mathcal{M}(k)\dot{\beta}_{\mathbb{B}}(k) + \overline{\mathcal{V}(k)}\mathcal{M}(k)\mathcal{V}(k) = \mathcal{F}(k) \quad (5)$$

where the spatial speed is projected, in this case, in the body frame $V(k) = \beta_{\mathbb{B}}(k)$. The external spatial forces considered are

$$\mathcal{F}(k) = \mathcal{F}_g(k) + \mathcal{F}_a(k) + \mathcal{F}_c(k) \quad (6)$$

$\mathcal{F}_g(k)$	body gravity force
$\mathcal{F}_a(k)$	body aerodynamic forces
$\mathcal{F}_c(k)$	rotor-body interaction force

C. Rotor Dynamics

Helicopter rotor dynamics is extremely complex, and is treated extensively in the literature [7] [1] [8] [9] [4], where the motion is studied in vertical and forward flight.

Many helicopters use Cierva articulated rotors with three hinges: feathering, flapping and lead-lag. There are helicopters with "hingeless" rotors (with only the feathering hinge) like the Cierva C.3, the Westland Lynx, the MBB Bo 105, or the Eurocopter EC135.

Since we deal with small electric helicopters, where the varying speed and the pitch of the blades can be used as controls, we model the rotor as a tilting propeller, like the one found in some autogyros.

A rotor is also modeled as a rigid body. So its dynamic equation is

$$\mathcal{M}_r(r)\dot{\beta}(r)+\overline{\mathcal{V}(r)}\mathcal{M}_r(r)\mathcal{V}(r) = \mathcal{F}_p(r)+\mathcal{F}_g(r)+\mathcal{F}_c(r) \quad (7)$$

identical to the body dynamics equation, where

$\mathcal{F}_p(r)$	aerodynamic rotor forces (thrust, torque)
$\mathcal{F}_g(r)$	gravity force
$\mathcal{F}_c(r)$	body-rotor interaction force

therefore the body-rotor interaction force is

$$\mathcal{F}_c(r) = \mathcal{M}_r(r)\dot{\beta}(r) + \overline{\mathcal{V}(r)}\mathcal{M}_r(r)\mathcal{V}(r) - \mathcal{F}_p(r) - \mathcal{F}_g(r)$$

and the rotor-body force, corrected by attitude and position, on the body frame reference origin k is

$$\mathcal{F}_c(k) = -\Phi(k,r)^k R_r(\epsilon)\mathcal{F}_c(r)V(K) \quad (8)$$

where

$\mathcal{F}_c(r)$	body-rotor action
$-\mathcal{F}_c(r)$	rotor-body reaction
${}^kR_r(\epsilon)$	rotation between frames of reference
$\phi(k,r)$	force transformation from point r to point k

therefore the force due to a rotor acting on the vehicle body is

$$\begin{aligned}\mathcal{F}_c(k) =&\Phi(k,r)^k R_r(\epsilon)\bigg(\mathcal{F}_p(r) + \mathcal{F}_g(r) - \mathcal{M}_r(r)\dot{\beta}(r) \\ &- \overline{\mathcal{V}(r)}\mathcal{M}_r(r)\mathcal{V}(r)\bigg) \quad (9)\end{aligned}$$

as it can be seen the action of the spatial aerodynamic forces of the propeller does not change from the simpler assumption that propeller has weight and only provides thrust and torque

$$\mathcal{F}_c(k) = \Phi(k,r)^k R_r(\epsilon)\bigg(\mathcal{F}_p(r) + \mathcal{F}_g(r)\bigg) \quad (10)$$

which are the usual spatial forces considered.

D. Spatial Speeds and Accelerations

The spatial speed and acceleration of a rotor of a helicopter with speed $\mathcal{V}(k)$, considering as the rotor axis of rotation z+ and rotation speed Ω

$$\begin{aligned}\mathcal{V}(r) =&{}^rR_k(\epsilon)\ \Phi^*(k,r)\mathcal{V}(k) + \dot{\epsilon}_r + \Omega_r \quad (11)\\ \dot{\beta}(r) =&{}^rR_k(\epsilon)\ \Phi^*(k,r)\dot{\beta}(k) - \tilde{\dot{\epsilon}}_r{}^rR_k(\epsilon)\Phi^*(k,r)\mathcal{V}(k) \\ &+ \ddot{\epsilon}_r + \dot{\Omega}_r - \tilde{\Omega}_r{}^rR_k(\epsilon)\Phi^*(k,r)\mathcal{V}(k) \quad (12)\\ {}^kR_r(\epsilon) =&R_x(\phi)R_y(\theta) \quad (13)\\ \dot{\epsilon}_r =&\{\dot{\phi}\cos\theta, \dot{\theta}, \dot{\phi}\sin\theta\}^T \quad (14)\end{aligned}$$

in this model, we consider that the rotor can tilt on the x-axis (roll angle ϕ) and on the y-axis (pitch angle θ) from the body, $\epsilon = \{\phi, \theta, 0\}^T$.

In a conventional helicopter, we have two rotors: tail rotor and main rotor. The tail rotor has a fixed orientation and is used to counteract the main rotor torque, enabling yaw control. The main rotor must tilt to provide, besides the lifting force, longitudinal and lateral forces (motion).

E. Body+Rotors Dynamic Equation

The general body-rotor force has the form

$$\begin{aligned}\mathcal{F}_c(r) =&\mathcal{M}_r(r) \\ &({}^rR_k(\epsilon)\ \Phi^*(k,r)\dot{\beta}(k) - \tilde{\dot{\epsilon}}_r{}^rR_k(\epsilon)\Phi^*(k,r)\mathcal{V}(k) \\ &+ \ddot{\epsilon}_r + \dot{\Omega}_r - \tilde{\Omega}_r{}^rR_k(\epsilon)\Phi^*(k,r)\mathcal{V}(k)) + \\ &\overline{({}^rR_k(\epsilon_r)\ \Phi^*(k,r)\mathcal{V}(k) + \dot{\epsilon}_r + \Omega_r)}\mathcal{M}_r(r) \\ &({}^rR_k(\epsilon_r)\ \Phi^*(k,r)\mathcal{V}(k) + \dot{\epsilon}_r + \Omega_r) \\ &- \mathcal{F}_p(r) - \mathcal{F}_g(r) \quad (15)\end{aligned}$$

therefore the helicopter+rotors dynamic equation becomes

$$\begin{aligned}\mathcal{M}(k)\dot{\beta}(k)+\overline{\mathcal{V}(k)}\mathcal{M}(k)\mathcal{V}(k) =& \mathcal{F}_g(k) + \mathcal{F}_a(k) \\ &- \sum_r \Phi(k,r)^k R_r(\epsilon)\mathcal{F}_c(r) \quad (16)\end{aligned}$$

expanding the rotor equation and rearranging terms, we have

$$\begin{aligned}&\bigg(\mathcal{M}(k) + \sum_r \Phi(k,r)^k R_r(\epsilon)\mathcal{M}_r(r)^r R_k(\epsilon)\Phi^*(k,r)\bigg)\dot{\beta}(k) \\ &+ \sum_r \Phi(k,r)^k R_r(\epsilon)\mathcal{M}_r(r)\bigg(- \tilde{\dot{\epsilon}}_r{}^rR_k(\epsilon)\Phi^*(k,r)\mathcal{V}(k) + \ddot{\epsilon}_r \\ &+ \dot{\Omega}_r - \tilde{\Omega}_r{}^rR_k(\epsilon)\Phi^*(k,r)\mathcal{V}(k)\bigg) + \\ &\overline{\mathcal{V}(k)}\mathcal{M}(k)\mathcal{V}(k) + \sum_r \Phi(k,r)^k R_r(\epsilon)\overline{\mathcal{V}(r)}\mathcal{M}_r(r)\mathcal{V}(r) = \\ &\mathcal{F}_g(k) + \mathcal{F}_a(k) + \sum_r \Phi(k,r)^k R_r(\epsilon)\bigg(\mathcal{F}_p(r) + \mathcal{F}_g(r)\bigg) \quad (17)\end{aligned}$$

so resulting direct dynamic equation for a helicopter is

$$\dot{\beta}(k) = \left(\mathcal{M}(k) + \sum_r \Phi(k,r)^k R_r(\epsilon)\mathcal{M}_r(r)^r R_k(\epsilon)\Phi^*(k,r)\right)^{-1} \Big[\mathcal{F}_{\mathbf{g}}(k) + \mathcal{F}_a(k) - \overline{\mathcal{V}(k)}\mathcal{M}(k)\mathcal{V}(k) + \sum_r \Phi(k,r)^k R_r(\epsilon) \Big\{\mathcal{F}_p(r) - \mathcal{M}_r(r)\Big(- \tilde{\dot{\epsilon}}_r{}^r R_k(\epsilon)\Phi^*(k,r)\mathcal{V}(k) + \ddot{\epsilon}_r + \dot{\Omega}_r - \tilde{\Omega}_r{}^r R_k(\epsilon)\Phi^*(k,r)\mathcal{V}(k)\Big) - \overline{\mathcal{V}(r)}\mathcal{M}_r(r)\mathcal{V}(r)\Big\}\Big] \quad (18)$$

III. Affine Control Form

The dynamic equation has the form

$$\dot{\beta}(k) = A^{-1}(\epsilon)F(\epsilon,\Omega) \quad (19)$$

with

$$A(\epsilon) = \mathcal{M}(k) + \sum_r \Phi(k,r)^k R_r(\epsilon)\mathcal{M}_r(r)^r R_k(\epsilon)\Phi^*(k,r) \quad (20)$$

$$F(\epsilon,\Omega) = \mathcal{F}_{\mathbf{g}}(k) + \mathcal{F}_a(k) - \overline{\mathcal{V}(k)}\mathcal{M}(k)\mathcal{V}(k) + \sum_r \Phi(k,r)^k R_r(\epsilon)\Big(\mathcal{F}_p(r) - \mathcal{M}_r(r)\Big(- \tilde{\dot{\epsilon}}_r{}^r R_k(\epsilon)\Phi^*(k,r)\mathcal{V}(k) + \ddot{\epsilon}_r + \dot{\Omega}_r - \tilde{\Omega}_r{}^r R_k(\epsilon)\Phi^*(k,r)\mathcal{V}(k)\Big) - \overline{\mathcal{V}(r)}\mathcal{M}_r(r)\mathcal{V}(r)\Big) \quad (21)$$

to get an affine form we must consider the variation due to ϵ and Ω

$$\dot{\beta}(k) = A^{-1}(\epsilon)F(\epsilon,\Omega) + \left(\frac{dA^{-1}(\epsilon)}{d\epsilon}F(\epsilon,\Omega) + A^{-1}(\epsilon)\frac{dF(\epsilon,\Omega)}{d\epsilon}\right)d\epsilon + A^{-1}(\epsilon)\frac{dF(\epsilon,\Omega)}{d\Omega}d\Omega \quad (22)$$

$$\dot{\epsilon} = \frac{1}{dt}d\epsilon \quad (23)$$

$$\dot{\Omega} = \frac{1}{dt}d\Omega \quad (24)$$

so $\{\epsilon,\Omega\}$ become part of the state and $\{d\epsilon, d\Omega\}$ part of the control. Using for the first term

$$\frac{dA^{-1}(\epsilon)}{d\epsilon} = -A^{-1}(\epsilon)\frac{dA(\epsilon)}{d\epsilon}A^{-1}(\epsilon) \quad (25)$$

where

$$\frac{dA}{d\epsilon} = \Phi(k,r)^k R_r(\epsilon)\Big(\tilde{\varepsilon}_r\mathcal{M}_r(r) - \mathcal{M}_r(r)\tilde{\varepsilon}_r^T\Big)^r R_k(\epsilon)\Phi^*(k,r) \quad (26)$$

this expression appears twice, due to the two control actions

$$\varepsilon_r(\phi) = \{\cos\theta, 0, \sin\theta\}^T \quad (27)$$

$$\varepsilon_r(\theta) = \{0,1,0\}^T \quad (28)$$

For the second term

$$\frac{dF(\epsilon,\Omega)}{d\epsilon} = \Phi(k,r)^k R_r(\epsilon)\tilde{\varepsilon}_r \Big(\mathcal{F}_p(r) - \mathcal{M}_r(r)\Big(- \tilde{\dot{\epsilon}}_r{}^r R_k(\epsilon)\Phi^*(k,r)\mathcal{V}(k) + \ddot{\epsilon}_r + \dot{\Omega}_r - \tilde{\Omega}_r{}^r R_k(\epsilon)\Phi^*(k,r)\mathcal{V}(k)\Big) - \overline{\mathcal{V}(r)}\mathcal{M}_r(r)\mathcal{V}(r)\Big) - \Phi(k,r)^k R_r(\epsilon)\mathcal{M}_r(r)\big(\tilde{\dot{\epsilon}}_r + \tilde{\Omega}_r\big)\tilde{\varepsilon}_r{}^r R_k(\epsilon)\Phi^*(k,r)\mathcal{V}(k) \quad (29)$$

and

$$\frac{dF(\epsilon,\Omega)}{d\Omega} = \Phi(k,r)^k R_r(\epsilon)\left(\frac{d\mathcal{F}_p(r)}{d\Omega} - \overline{\frac{d\mathcal{V}(r)}{d\Omega}}\mathcal{M}_r(r)\mathcal{V}(r) - \overline{\mathcal{V}(r)}\mathcal{M}_r(r)\frac{d\mathcal{V}(r)}{d\Omega}\right) \quad (30)$$

$$\frac{d\mathcal{V}(r)}{d\Omega} = \{0,0,1,0,0,0\}^T \quad (31)$$

where $\frac{d\mathcal{F}_p(r)}{d\Omega}$ will be calculated in the next section

IV. Rotor Forces

A large part of the literature deals with the rotor forces. The flapping and feathering movements are considered in detail. In the previous equations $\mathcal{F}_p(r)$ In this work we will first determine the local spatial speed associated with a blade element, from it we will calculate the local spatial forces. The spatial speed at the rotation axis is

$$\mathcal{V}(r) = {}^r R_k(\epsilon_r)\ \Phi^*(k,r)\mathcal{V}(k) + \dot{\epsilon}_r + \Omega_r \quad (32)$$

the point r is at the origin of the frame of reference with axes $x-y$ on the rotor rotation plane. On this plane there are $n=3$ blades at angles $\psi_1 = \psi(t), \psi_2 = \psi(t) + 2\pi/3, \psi_3 = \psi(t) + 4\pi/3$. With

$$\psi(t) = \int_0^t \Omega(t)dt \quad (33)$$

the local speed at a point of radio y of blade $i = \{1,\dots,n\}$ is

$$\mathcal{V}(y) = \Phi^*(r,y)R_z(\psi_i)\left({}^r R_k(\epsilon_r)\ \Phi^*(k,r)\mathcal{V}(k) + \dot{\epsilon}_r + \Omega_r\right) \quad (34)$$

considering the flow perpendicular to the blade (y pointing axis lengthwise), we can calculate the average force per revolution

$$\mathcal{F}_i(r) = \frac{1}{2\pi}\frac{1}{2}\rho\,\bar{c} \int_0^{2\pi}\int_0^R \left(\mathtt{u}_\mathtt{r}^2 + \mathtt{w}_\mathtt{r}^2\right) {}^r R_i(\psi_i)\,\Phi(r,y)\,C_F\,dy\,d\psi_i \quad (35)$$

with

$$\alpha = \arctan(\mathtt{w_r}/\mathtt{u_r}) \tag{36}$$

$$\alpha_r = \alpha + \iota \tag{37}$$

$$C_F = \begin{bmatrix} 0 \\ C_m \\ 0 \\ -C_D \cos(\alpha_r) + C_L \sin(\alpha_r) \\ 0 \\ -C_D \sin(\alpha_r) - C_L \cos(\alpha_r) \end{bmatrix} \tag{38}$$

being α the disk local angle of attack and ι the collective angle.

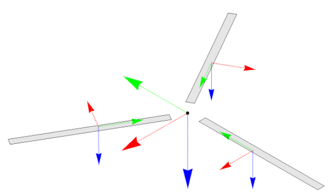

Fig. 2. Rotor blade element local frames, the color code for the axes is (x-y-z)→(red-green-blue)

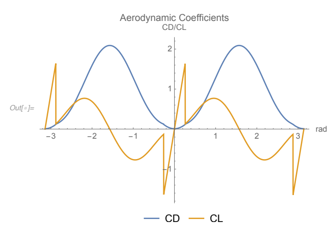

Fig. 3. No-linear behavior of the aerodynamic coefficients consider in the model $\alpha \in [-\pi, \pi]$, a similar graph is found in [10]

The total spatial force due to the rotor is the sum of the blades individual forces

$$\mathcal{F}_p(r) = \sum_i \mathcal{F}_i(r) \tag{39}$$

and the control actions are the change in the disk attitude $\epsilon = \{\phi, \theta\}$, the collective angle ι and the speed Ω

$$\frac{d\mathcal{F}_p(r)}{d*} = \sum_i \frac{d\mathcal{F}_i(r)}{d*} \tag{40}$$

to simplify the computations, a point at the midspan of the blade is used for the calculus of the whole blade force.

V. Simulation

Before doing simulations we would have liked to derive a linear model from the nonlinear and draw the matrix control flow diagram (MCFD) of the system. These items would require an analysis of the equilibrium of the vehicle. Yet having the model, debugging it is the first step in this path. In order to debug the model a simple simulation scheme was implemented.

Here we present preliminary results of the vehicle simulation under a feedback linearization scheme for hovering, constant speed rotors, and modified Rodrigues parameter for attitude specification (that allow us to avoid singularities). The simulated vehicle has a 1 800 kg mass, 10.5 m main rotor diameter and 0.7 m tail rotor diameter. However, we expect to work in small RC vehicles. A simulation of 1 s takes almost 50 s in our computer, with a step of 0.01 s. The use of Mathematica's interpreter makes the model not easy to debug, at the time of the simulation the performance is far from what we desire. Several outputs were considered but so far the best results are with the output presented, from the magnitud of the control, it can be seen that the system is extremely sensitive.

This is an ongoing research, and the main purpose of the article is to present the dynamics of a three body model. A simpler model might get us better results, e.g. ignoring the tail rotor torque, or considering the system as one body or a three degree of freedom vehicle or simpler aerodynamic forces or a more robust control scheme. Experiments will have to be done with some of this simpler assumptions, to understand the interactions of the model; but if the end is to control a real vehicle whose behavior is potentially dangerous to its environment and the only way we see that we can be achieved this, starting from near complete ignorance, is to develop a as much as possible complete model. That still does not include all the phenomena that occur in this kind of vehicles, e.g., the rotor blades flapping, where each blade dynamics can be modeled using the same methodology. The aspect ratio of the small RC blades, near 5, might make this phenomenon less likely.

VI. Conclusions

There are many models for helicopters and similar vehicles in the literature, a superficial review of texts in the area [11] [1] [8] [12] [4] [13] yields models based on the Newton-Euler equations. A very complete model is found in the work of Talbot et al. [10]. An issue discussed in extent in the literature is the flexibility of the rotor blades. Here, we assume rigid elements, therefore, one piece rigid rotors. Maybe the biggest flaw of our model.

The Newton-Euler dynamic equations have been the standard in the development of the dynamic equations, since [14]. The main difference of our model from others found in the literature, is the use of spatial operator algebra that allow us to model the joint linear and rotational dynamic of the vehicle as a chain of interconnected links. In our case, we consider the vehicle formed by three links: the body, the main rotor and the tail rotor. We are not restricted to the center of gravity as the reference point to write the equations, any point of the vehicle can be used. The second Newton law is use to calculate the interaction between bodies. A similar result, not presented here, is obtained using the algorithms for serial-chain multibodies.

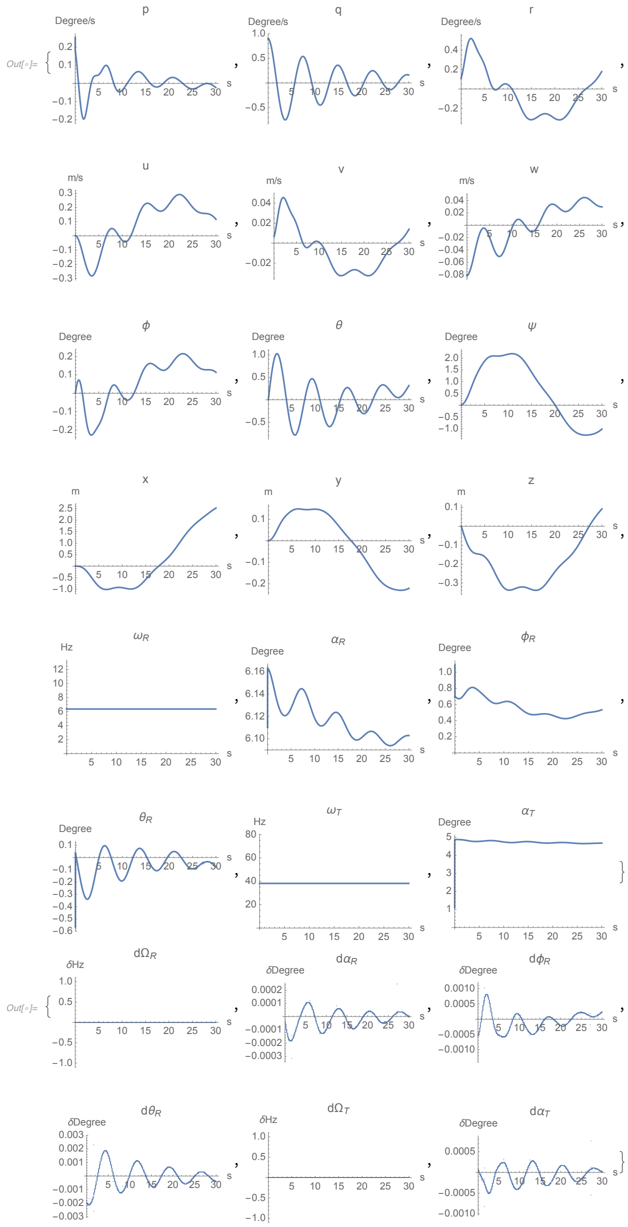

Fig. 4. Rotary wing vehicle state and control: spatial speed $\{p, q, r, u, v, w\}$, euler angles attitude $\{\phi, \theta, \psi\}$ (converted from the modified Rodrigues parameters attitude), position $\{x, y, z\}$, main rotor speed, collective angle, roll and pitch angles $\{\omega_R, \alpha_R, \phi_R, \theta_R\}$, $\{d\Omega_R, d\alpha_R, d\phi_R, d\theta_R\}$, tail rotor speed and collective angle $\{\omega_T, \alpha_T\}$, $\{d\Omega_T, d\alpha_T\}$

An important part of the model is the calculus of the aerodynamic forces on both rotors, main and tail, under any air flow condition, that can render very asymmetric generalized forces. This is a computationally expensive model, and we opted to not consider the instantaneous rotor forces, but the average force on a revolution of one blade and multiplying it by the number of blades. Further study on the subject is in process.

The resulting model is not affine in the control. Therefore, an affine control form must be obtained considering the effect of the change on the main rotor tilt ($\epsilon = \{\phi_r, \theta_r\}$), both rotors speed ($\Omega$) and collective blade pitch (ι) on the vehicle dynamics. This model has been programmed in Mathematica and will allow the use of standard control algorithms.

A main characteristic of any helicopter is its hovering capability. The model help us to find the conditions under which this is possible.

This is our first model of a rotary wing aircraft, it is work in progress. In this work the emphasis is on the interaction between the bodies that form the rotary wing aircraft, how the air flow reaches the rotors and how to have a system affine in the control from a system that is not.

In nature, rotating "wings" are called *samara*, the generic name for winged fruit or seed that autorotate when they fall, e.g. Maple seeds.

APPENDIX

The bar operator used in the model is defined by

$$\bar{Z} = \begin{bmatrix} \tilde{x} & \tilde{y} \\ 0 & \tilde{x} \end{bmatrix} \in \mathbb{R}^{6\times 6} \tag{41}$$

for an spatial vector $Z = \{x, y\} \in \mathbb{R}^6$, with

$$\tilde{x} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \tag{42}$$

for $x = \{x_1, x_2, x_3\}^T \in \mathbb{R}^3$ The speed at a point y at the rotation plane of a blade

$$\mathcal{V}(y) = \{\mathtt{p_r}, \mathtt{q_r}, \mathtt{r_r}, \mathtt{u_r}, \mathtt{v_r}, \mathtt{w_r}\}^T \tag{43}$$

the change on the rotor force due to a control action $* \in \{\epsilon, \iota, \Omega\}$

$$\frac{\partial \mathcal{F}_i(r)}{\partial *} = \frac{1}{2\pi}\frac{1}{2}\rho\bar{c}\Bigg(\int_0^{2\pi}\int_0^R \left(2\mathtt{u_r}\frac{d\mathtt{u_r}}{d*} + 2\mathtt{w_r}\frac{d\mathtt{w_r}}{d*}\right){}^rR_i(\psi_i)\Phi(r,y)C_F\,dy\,d\psi_i + \int_0^{2\pi}\int_0^R \left(\mathtt{u_r}^2 + \mathtt{w_r}^2\right){}^rR_i(\psi_i)\Phi(r,y)\frac{dC_F}{d*}\,dy\,d\psi_i\Bigg) \tag{44}$$

$$\frac{d\mathcal{V}(x)}{d\epsilon} = \Phi^*(r,x)R_z(\psi_i)^rR_k(\epsilon_r)\tilde{\varepsilon}\Phi^*(k,r)\mathcal{V}(k) \tag{45}$$

$$\frac{d\mathcal{V}(x)}{d\Omega} = \Phi^*(r,x)R_z(\psi_i)\begin{bmatrix}0 & 0 & 1 & 0 & 0 & 0\end{bmatrix}^T \tag{46}$$

$$\frac{dC_F}{d\alpha_r} = \begin{bmatrix} 0 \\ \frac{dC_m}{d\alpha_r} \\ 0 \\ -\frac{dC_D}{d\alpha_r}\cos(\alpha_r) + \frac{dC_L}{d\alpha_r}\sin(\alpha_r) \\ 0 \\ -\frac{dC_D}{d\alpha_r}\sin(\alpha_r) - \frac{dC_L}{\alpha_r}\cos(\alpha_r) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ C_D\sin(\alpha_r) + C_L\cos(\alpha_r) \\ 0 \\ -C_D\cos(\alpha_r) + C_L\sin(\alpha_r) \end{bmatrix} \quad (47)$$

$$\frac{dC_F}{d*} = \frac{dC_F}{d\alpha_r}\frac{d\alpha_r}{d*} \quad (48)$$

$$\frac{d\alpha_r}{d*} = \frac{d\alpha_r}{d\alpha}\frac{d\alpha}{d*} = \frac{1}{\mathtt{u}_\mathtt{r}^2 + \mathtt{w}_\mathtt{r}^2}(\mathtt{u}_\mathtt{r}\frac{d\mathtt{w}_\mathtt{r}}{d*} - \mathtt{w}_\mathtt{r}\frac{d\mathtt{u}_\mathtt{r}}{d*}) \quad (49)$$

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