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Simple Optimal Tracking Control for a Class of Closed-Chain Mechanisms in Task Space

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***Abstract*—In this work an optimal tracking control in task space for slider-crank mechanisms is presented. To guarantee the tracking task it is required the complete knowledge of mechanism non-linear dynamics for the optimal control design, however the complete dynamics is not always available and linearization methods may lose many information.**

From the extended dynamic model of parallel robots, it can be seen that a linear system based on the slider dynamics can be obtained without any kind of linearization. This approach avoids the complete knowledge of the mechanism dynamics. Experiments are made to verify and compare the performance of the simplified model with the complete linearized dynamics model using the linear quadratic tracking (LQT) and sliding mode control (SMC).

I. Introduction

Tracking control of non-linear mechanical systems (especially robot manipulators and mechanisms) is a well known control problem where a control law is designed to achieve a desired trajectory tracking. To obtain an optimal performance, the controller is designed using optimal control theory. This is a well known discipline that finds optimal controllers for dynamic systems by optimizing a certain cost function or index. The control law needs the complete knowledge of the non-linear mechanical system [1] to design a new system in function of the desired trajectory. If the non-linear system is not available, the control law rely on some approximations and/or linearizations [2].

There exist several methods to linearize robot/mechanism models. The simplest method is when velocity and gravity are neglected, then the Coriolis matrix and the gravity forces vector are zero [3], however this is an oversimplified model [4]. If gravity is taken into account, it is obtained another way to linearize the dynamics [5], but it has been shown that the Coriolis effect even at low speeds should be accounted for [6]. The most used method is Taylor series expansion [7], but requires the knowledge of the robot dynamics model. All the above methods make the linearization at one operating point or equilibrium point, then the controller is restricted to areas near that point. In the special case of mechanisms, the linearization gives an oversimplified model and many information is lost since all the generalized coordinates depend on the independent coordinate.

To satisfy the control objectives there are developed different controllers, such as PD+ [8], PID [9], [10], adaptive [11], [12], sliding mode [12]–[14], neural networks [15], among others [2]. However the gains are tuned manually in most cases and do not guarantee an optimal performance of the closed-loop system. PID control is the most popular controller for industrial applications. The controller gains tuning require knowledge of the complete dynamics and a linear model at a point of interest. However the integral term reduces the bandwidth and can destroy the closed-loop performance [12]. Other controller widely used is sliding mode control (SMC) [13], [14] where the tracking control is guaranteed by choosing a large gain, however the sliding hyperplane gain is chosen arbitrarily because the mechanism dynamics is non-linear.

Since the aim of this paper is to guarantee an optimal trajectory for closed-chain mechanisms, we need to know the complete mechanism dynamics which is not always available. If the model presents modeling error, then the closed-loop performance loses accuracy and does not achieve the tracking task. Therefore, most of the authors sacrifice the optimal performance to guarantee accuracy at the tracking task. However there exists a class of closed-chain mechanisms that has a linear movement at its end-effector, slider crank mechanisms [16]. This mechanisms are wide used for machine tools or molding machines [17] for cutting tasks. One of the most used slider crank mechanism is the Whitworth mechanism [11], [17], [18] which is of our interest. Generally the Whitworth mechanism is controlled in joint space [19] but its preferable to control it in task space because its end-effector has a linear movement. By using this fact, we can obtain an optimal tracking control for this class of mechanisms without knowledge of the complete mechanism dynamics, space transformation (from joint space to task space) and linearization methods.

This work proposes an optimal tracking control in task space for a class of closed-chain mechanisms (slider-crank mechanisms) without knowledge of the complete dynamics and linearization methods. From the end-effector linear movement and the extended dynamic model is obtained a linear model which depends only on the end-effector or slider mass. This model facilitate the whole control design with a minimal knowledge of the dynamics. The outline of the paper is as follows: Section II shows the dynamic model of slider-crank

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mechanisms and the optimal tracking control design in a non-linear and linear cases, and sliding mode. Section III gives the extended dynamic model and our model approach. Section IV shows the experimental results using an inverted Whitworth mechanism prototype and Section V concludes the paper.

II. Dynamic model of a 1-DOF closed-chain slider crank mechanisms

The dynamic model [2] in joint space of a 1-DOF closed-chain slider crank mechanism is of the form:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau, \quad (1)$$

where $M(q) \in \mathbb{R}$ is the mechanism inertia, $C(q,\dot{q}) \in \mathbb{R}$ is the Coriolis term, $G(q) \in \mathbb{R}$ is the gravitational term, $\tau \in \mathbb{R}$ is the driven torque and $q, \dot{q}, \ddot{q} \in \mathbb{R}$ are the position, velocity and acceleration of the generalized coordinate. In order to transform the model (1) to task space [21] it is required to use the Jacobian term $\rho_x(q) \in \mathbb{R}$ of the slider component as:

$$M_x\ddot{x} + C_x\dot{x} + G_x = \rho_x^{-1}(q)\tau = u, \quad (2)$$

where $x, \dot{x}, \ddot{x} \in \mathbb{R}$ are the position, velocity and acceleration of the slider. The terms M_x, C_x and G_x are obtained from the velocity kinematics relation $\dot{x} = \rho_x(q)\dot{q}$. The model (2) in state space is expressed as

$$\dot{\boldsymbol{z}}(t) = \frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \underbrace{\begin{bmatrix} z_2 \\ -M_x^{-1}(C_x z_2 + G_x) \end{bmatrix}}_{\boldsymbol{f(z)}} + \underbrace{\begin{bmatrix} 0 \\ M_x^{-1} \end{bmatrix}}_{\boldsymbol{g(z)}} u \quad (3)$$

It is assumed that $\boldsymbol{f(0)} = \boldsymbol{0}$ and $\boldsymbol{f(z)} + \boldsymbol{g(z)}u(t)$ is locally Lipschitz.

A. Optimal Tracking Problem: Non-linear case

The main goal is to design an optimal control input to make the states of the system $\boldsymbol{z}(t) \in \mathbb{R}^2$ follow a desired reference trajectory $\boldsymbol{z_d}(t) \in \mathbb{R}^2$ [20]. Let define the tracking error $\boldsymbol{e}(t) \in \mathbb{R}^2$ as

$$\boldsymbol{e}(t) = \boldsymbol{z}(t) - \boldsymbol{z_d}(t).$$

In a tracking problem, the control input consists of two terms: a feedforward term that guarantees the position tracking and a feedback term that stabilizes the system.

The feedforward term can be obtained using the inverse of the dynamics in function of the desired reference as

$$u_d(t) = \boldsymbol{g}^{-1}(\boldsymbol{z}(t))\left(\dot{\boldsymbol{z}}_{\boldsymbol{d}}(t) - \boldsymbol{f}(\boldsymbol{z_d}(t))\right). \quad (4)$$

The closed loop dynamics between (3) and the feedforward control (4) is:

$$\dot{\boldsymbol{e}}(t) = \boldsymbol{f}(\boldsymbol{z}) - \boldsymbol{f}(\boldsymbol{z_d}) \leq L\|\boldsymbol{z}(t) - \boldsymbol{z_d}(t)\| = L\|\boldsymbol{e}(t)\|, \quad (5)$$

where $L > 0$ is a Lipschitz constant. Then to stabilize the closed-loop is required to design a feedback control as

$$\dot{\boldsymbol{e}}(t) = \boldsymbol{f}(\boldsymbol{z}) - \boldsymbol{f}(\boldsymbol{z_d}) + \boldsymbol{H}u_e(t),$$

where $\boldsymbol{H} = [0,1]^\top$ and $u_e(t)$ is the feedback control. In this work, the feedback gain is obtained my minimizing the following cost function:

$$J(\boldsymbol{e}(t), u_e(t)) = \int_t^\infty \left((\boldsymbol{e}^\top(\sigma)\boldsymbol{Q}\boldsymbol{e}(\sigma) + Ru_e^2(\sigma)\right) d\sigma \quad (6)$$

where $\boldsymbol{Q} \succeq 0$ and $R > 0$ are the weights of the tracking error and the feedback control. The feedback input can be obtained by applying $\partial J(\boldsymbol{e}, u_e)/\partial u_e = 0$ as:

$$u_e^*(t) = -\frac{1}{2R}\boldsymbol{H}^\top \frac{\partial J(\boldsymbol{e}, u_e)}{\partial \boldsymbol{e}}. \quad (7)$$

Then, the optimal control input that includes both feedforward and feedback terms is

$$u^*(t) = u_d(t) + \boldsymbol{g}^{-1}(\boldsymbol{z}(t))\boldsymbol{H}u_e^*(t). \quad (8)$$

B. Optimal Tracking Problem: Linear case

The optimal tracking problem of the last section requires knowledge of the non-linear dynamics to design the controller, however it is a hard problem especially the feedback term. In order to simplify the controller design, some authors linearize the dynamics in an operating point. The linear version of the dynamics is given as follows

$$\dot{\boldsymbol{z}}(t) = \boldsymbol{A}\boldsymbol{z}(t) + \boldsymbol{B}u(t), \quad (9)$$

where $\boldsymbol{A} = \frac{\partial \boldsymbol{f(z)}}{\partial \boldsymbol{z}}\Big|_{\boldsymbol{z}(t)=\boldsymbol{z}(0)}$ and $\boldsymbol{B} = \frac{\partial \boldsymbol{g(z)}}{\partial u}\Big|_{u(t)=u(0)}$ and the pair $(\boldsymbol{A}, \boldsymbol{B})$ is controllable, here $\boldsymbol{z}(0)$ and $u(0)$ are the linearization points.

The feedforward control is designed as

$$u_d(t) = \boldsymbol{B}^+\left(\dot{\boldsymbol{z}}_{\boldsymbol{d}}(t) - \boldsymbol{A}\boldsymbol{z_d}(t)\right), \quad (10)$$

where $\boldsymbol{B}^+$ is the Moore-Penrose pseudo-inverse of $\boldsymbol{B}$. The closed-loop dynamics under the feedforward control is

$$\dot{\boldsymbol{e}}(t) = \boldsymbol{A}\boldsymbol{e} + \boldsymbol{H}u_e(t)$$

The cost function is quadratic in the current state as:

$$J(\boldsymbol{e}(t)) = \boldsymbol{e}^\top(t)\boldsymbol{P}\boldsymbol{e}(t), \quad \forall \boldsymbol{e} \quad (11)$$

for some kernel matrix $\boldsymbol{P} = \boldsymbol{P}^\top$. Using (7) with (11) yields the optimal tracking control as:

$$u_e^*(t) = -\frac{1}{R}\boldsymbol{H}^\top\boldsymbol{P}\boldsymbol{e}(t). \quad (12)$$

The matrix $\boldsymbol{P}$ is obtained by solving the Algebraic Riccati Equation (ARE)

$$\boldsymbol{A}^\top\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} - \frac{1}{R}\boldsymbol{P}\boldsymbol{H}\boldsymbol{H}^\top\boldsymbol{P} + \boldsymbol{Q} = \boldsymbol{0}. \quad (13)$$

The optimal control input is:

$$u^*(t) = \boldsymbol{B}^+\left(\dot{\boldsymbol{z}}_{\boldsymbol{d}}(t) - \boldsymbol{A}\boldsymbol{z_d}(t) - \frac{1}{R}\boldsymbol{H}^\top\boldsymbol{P}\boldsymbol{e}(t)\right). \quad (14)$$

The above controller guarantees the tracking task for the linear system (9), nevertheless the controller cannot be applied directly at the non-linear dynamics since the linearization gives an oversimplified model that does not take into account some non-linearities of the mechanism dynamics.

C. Optimal Sliding mode control

The main problem of the optimal tracking problem is the feedforward term. If the dynamics of the mechanism is unknown or has modeling error, then the tracking task cannot be guaranteed and can lead to a large position error.

One way to obtain an optimal performance without knowledge of the dynamics is by means of sliding mode control (SMC). The controller is designed to force the system to the switching hyperplane:

$$s(\boldsymbol{e}(t)) = \boldsymbol{C}\boldsymbol{e}(t), \tag{15}$$

where $\boldsymbol{C}$ is the switching gain. Since we want to avoid the feedforward term, the closed-loop dynamics is rewritten as:

$$\dot{\boldsymbol{e}}(t) = \boldsymbol{A}\boldsymbol{e}(t) + \boldsymbol{B}\left(u(t) + \boldsymbol{B}^{+}\boldsymbol{D}(\boldsymbol{z}_d(t))\right), \tag{16}$$

where $\boldsymbol{D}(\boldsymbol{z}_d(t)) = \boldsymbol{A}\boldsymbol{z}_d - \dot{\boldsymbol{z}}_d$. To facilitate the design of the switching hyperplane consider the following coordinate transformation

$$\boldsymbol{\Psi}(t) = \boldsymbol{T}\boldsymbol{e}(t),$$

where $\boldsymbol{T} = \left[\boldsymbol{B}^{\perp^{+}}, \boldsymbol{B}^{+}\right]^{\top}$. The system under the new coordinates is

$$\dot{\boldsymbol{\Psi}}(t) = \underbrace{\boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1}}_{\bar{\boldsymbol{A}}}\boldsymbol{\Psi}(t) + \underbrace{\boldsymbol{T}\boldsymbol{B}}_{\bar{\boldsymbol{B}}}\left(u(t) + \boldsymbol{B}^{+}\boldsymbol{D}(\boldsymbol{z}_d(t))\right),$$

where $\bar{\boldsymbol{A}} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $\bar{\boldsymbol{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Hence the new system is

$$\begin{aligned} \dot{\Psi}_1(t) &= A_{11}\Psi_1(t) + A_{12}\Psi_2(t) \\ \dot{\Psi}_2(t) &= A_{21}\Psi_1(t) + A_{22}\Psi_2(t) + u(t) + d(t), \end{aligned} \tag{17}$$

where $\boldsymbol{B}^{+}\boldsymbol{D}(\boldsymbol{z}_d(t)) = [0, d(t)]^{\top}$. The switching hyperplane (15) can be rewritten in the new coordinates as

$$s(\boldsymbol{\Psi}(t)) = \boldsymbol{C}\boldsymbol{T}^{-1}\boldsymbol{\Psi}(t) = C_1\Psi_1(t) + C_2\Psi_2(t) = \bar{\boldsymbol{C}}\boldsymbol{\Psi}(t).$$

Without loss of generality, we will assume that $C_2 = 1$ and hence

$$s(\boldsymbol{\Psi}(t)) = C_1\Psi_1(t) + \Psi_2(t). \tag{18}$$

The existence of a sliding mode implies $s(\boldsymbol{\Psi}(t)) = \dot{s}(\boldsymbol{\Psi}(t)) = 0$ for all $t > t_s$, where t_s is the finite time where the system trajectories converge to the hyperplane. The equivalent control is:

$$u_{eq} = -\left[(C_1A_{11} + A_{21})\Psi_1 + (C_1A_{12} + A_{22})\Psi_2 + d\right],$$

which yields the equivalent system

$$\dot{\Psi}_1(t) = A_{11}\Psi_1(t) + A_{12}\Psi_2(t) \tag{19}$$

$$\begin{aligned} \dot{\Psi}_2(t) &= -C_1\left(A_{11}\Psi_1(t) + A_{12}\Psi_2(t)\right) \\ &= -C_1\dot{\Psi}_1(t), \end{aligned} \tag{20}$$

where the second equation satisfy the constraint equations $s(\boldsymbol{\Psi}(t)) = \dot{s}(\boldsymbol{\Psi}(t)) = 0$. Notice that $\Psi_2(t)$ plays the role of a virtual control input that we require to design in order to obtain the sliding gain C_1. We rely on the following well known result [22]:

Lemma 1: If the pair $(\boldsymbol{A}, \boldsymbol{B})$ of (9) is controllable, then the pair (A_{11}, A_{12}) of (19) is also controllable.

By using the above lemma in the equation (19), we can determine the optimal sliding surface which minimizes the following index

$$J(\boldsymbol{e}(t)) = \int_t^{\infty} \boldsymbol{e}^{\top}(\sigma)\boldsymbol{Q}\boldsymbol{e}(\sigma)d\sigma. \tag{21}$$

Note that the above index does not have control term because the sliding mode is independent of the control. In terms of the new coordinates yields

$$J(\boldsymbol{\Psi}(t)) = \int_t^{\infty} \left(\Psi_1^{\top}Q_{11}\Psi_1 + 2\Psi_1^{\top}Q_{12}\Psi_2 + \Psi_2^{\top}Q_{22}\Psi_2\right)d\sigma, \tag{22}$$

where $\boldsymbol{T}\boldsymbol{Q}\boldsymbol{T}^{-1} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$. However the index (22) has not the same form as the index (6). We use the backstepping method [23], where it is proposed the following new variable:

$$v(t) = \Psi_2(t) + Q_{22}^{-1}Q_{12}^{\top}\Psi_1(t) \tag{23}$$

Then the equation (19) and the index (22) are rewritten as

$$\dot{\Psi}_1(t) = \left(A_{11} - A_{22}Q_{22}^{-1}Q_{12}^{\top}\right)\Psi_1 + A_{12}v(t) \tag{24}$$

$$\begin{aligned} J(\boldsymbol{\Psi}) &= \int_t^{\infty} \left[\Psi_1^{\top}\left(Q_{11} - Q_{12}Q_{22}^{-1}Q_{12}^{\top}\right)\Psi_1 + v^{\top}Q_{22}v\right]d\sigma \\ &= \int_t^{\infty} \left(\Psi_1^{\top}(\sigma)\bar{Q}\Psi_1(\sigma) + v^{\top}(\sigma)Q_{22}v(\sigma)\right)d\sigma, \end{aligned} \tag{25}$$

where $\bar{Q} = Q_{11} - Q_{12}Q_{22}^{-1}Q_{12}^{\top}$. Note that (25) has now the same form as (6), then the optimal virtual control [23] is given by

$$v(t) = -Q_{22}^{-1}A_{12}^{\top}P\Psi_1(t). \tag{26}$$

Therefore the optimal sliding hyperplane is given by

$$\begin{aligned} \Psi_2(t) &= -C_1\Psi_1(t) \\ &= -Q_{22}^{-1}\left(A_{12}^{\top}P + Q_{12}^{\top}\right)\Psi_1(t). \end{aligned} \tag{27}$$

We only have to design the sliding mode controller. Consider the Lyapunov function

$$V(s(t)) = \frac{1}{2}s^2(t).$$

The time derivative of the Lyapunov function is

$$\begin{aligned} \dot{V}(s(t)) &= s\bar{\boldsymbol{C}}\dot{\boldsymbol{\Psi}}(t) \\ &= s\bar{\boldsymbol{C}}\left[\bar{\boldsymbol{A}}\Psi(t) + \bar{\boldsymbol{B}}\left(u(t) + \boldsymbol{B}^{+}\boldsymbol{D}(\boldsymbol{z}_d(t))\right)\right] \\ &= s\left[\bar{\boldsymbol{C}}\bar{\boldsymbol{A}}\boldsymbol{\Psi}(t) + u(t) + \boldsymbol{B}^{+}\boldsymbol{D}(\boldsymbol{z}_d(t))\right] \end{aligned}$$

By choosing $u(t) = -K\text{sign}(s)$, where $\text{sign}(\cdot)$ is the signum function and K is the controller's gain. By substituting the control into the time derivative of the Lyapunov function yields

$$\begin{aligned} \dot{V}(s(t)) &= s\left[\bar{\boldsymbol{C}}\bar{\boldsymbol{A}}\boldsymbol{\Psi}(t) - K\text{sign}(s) + \boldsymbol{B}^{+}\boldsymbol{D}(\boldsymbol{z}_d(t))\right] \\ &\leq -\|s(t)\|\left[K - \|\bar{\boldsymbol{C}}\bar{\boldsymbol{A}}\boldsymbol{\Psi}(t) + \boldsymbol{B}^{+}\boldsymbol{D}(\boldsymbol{z}_d(t))\|\right], \end{aligned}$$

by choosing $K = \|\bar{\boldsymbol{C}}\bar{\boldsymbol{A}}\boldsymbol{\Psi}(t) + \boldsymbol{B}^{+}\boldsymbol{D}(\boldsymbol{z}_d(t))\| + K_0$, yields

$$\dot{V}(s(t)) \leq -K_0\|s(t)\|.$$

III. Extended Dynamic model

The model (2) does not consider the main advantage of slider crank mechanisms, i.e., the linear movement of the slider. The transformation from joint space to task space considers all the dynamics from the input link to the output link and yields a more complex model in comparison to the joint space model (1), and if the dynamics is linearized at an operating point yields an oversimplified model that does not serve for control purposes. Also the models (1) and (2) require knowledge of the complete dynamics and mainly its parameters for the controller design.

In order to avoid the model (2) and linearization methods it is used the Euler-Lagrange formulation [24] for the extended dynamic model. The extended dynamic model of a 1-DOF closed-chain slider crank mechanism is of the form:

$$\boldsymbol{M}'(\boldsymbol{q}')\ddot{\boldsymbol{q}}' + \boldsymbol{C}'(\boldsymbol{q}', \dot{\boldsymbol{q}}')\dot{\boldsymbol{q}}' + \boldsymbol{G}'(\boldsymbol{q}') = \boldsymbol{B}_\tau \tau \tag{28}$$

where $\boldsymbol{q}' \in \mathbb{R}^{n'}$ are the extended coordinates whose components are the generalized coordinate q and all the n secondary variables. Here $n' = n+1$. $\boldsymbol{M}'(\boldsymbol{q}') \in \mathbb{R}^{n' \times n'}$ is the inertia matrix, $\boldsymbol{C}'(\boldsymbol{q}', \dot{\boldsymbol{q}}') \in \mathbb{R}^{n' \times n'}$ represent the centrifugal and Coriolis term, $\boldsymbol{G}'(\boldsymbol{q}') \in \mathbb{R}^{n'}$ is the gravity vector, $\tau \in \mathbb{R}$ is the control input and $\boldsymbol{B}_\tau = [1, \boldsymbol{0}]^\top \in \mathbb{R}^{n'}$. Notice that (28) is presented with the generalized coordinates $\boldsymbol{q}'$ instead of using the independent coordinate q. The extended dynamics is obtained by using the relation

$$\dot{\boldsymbol{q}}' = \boldsymbol{\rho}(\boldsymbol{q}')\dot{q},$$

where $\boldsymbol{\rho}(\boldsymbol{q}') \in \mathbb{R}^{n'}$ is the extended Jacobian vector. For slider crank mechanisms, the position of the slider, x, is a component of the generalized coordinate vector, i.e. $\boldsymbol{q}' = [q \cdots x]^\top$, that corresponds to the end-effector position which is the main control objective. In matrix form we have:

$$\begin{aligned}
\boldsymbol{M}'(\boldsymbol{q}') &= \begin{bmatrix} \boldsymbol{M}_{(n'-1)\times(n'-1)} & \boldsymbol{0}_{n'-1} \\ \boldsymbol{0}_{1\times(n'-1)} & m \end{bmatrix} \\
\boldsymbol{C}'(\boldsymbol{q}', \dot{\boldsymbol{q}}') &= \begin{bmatrix} \boldsymbol{C}_{(n'-1)\times(n'-1)} & \boldsymbol{0}_{n'-1} \\ \boldsymbol{0}_{1\times(n'-1)} & 0 \end{bmatrix} \\
\boldsymbol{G}'(\boldsymbol{q}') &= \begin{bmatrix} \boldsymbol{G}_{n'-1} \\ g_x \end{bmatrix}, \quad \boldsymbol{\rho}(\boldsymbol{q}') = \begin{bmatrix} \boldsymbol{\rho}_{n'-1} \\ \rho_x(\boldsymbol{q}') \end{bmatrix}
\end{aligned} \tag{29}$$

where m is the slider mass, g_x is the gravity force component of the slider that depends on the mechanism configuration, and $\rho_x(\boldsymbol{q}')$ is the Jacobian component that gives the mapping between the joint velocity $\dot{q}$ and the slider velocity $\dot{x}$. For a task position control problem, we want to control the slider position to a desired position, then the control problem simplifies to:

$$m\ddot{x}(t) + g_x = 0,$$

the above model only requires the knowledge of the slider mass and avoids the knowledge of the complete mechanism dynamics. Since we can relate the torque input with the slider dynamics using the Jacobian of the slider component as, i.e., $\tau = \rho_x^{-1}(\boldsymbol{q}')u$, then the slider dynamics is rewritten as

$$m\ddot{x}(t) + g_x = u(t) \tag{30}$$

It is clear that (30) is a linear system with a disturbance. In state space we have:

$$\dot{\boldsymbol{z}}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\boldsymbol{A}} \boldsymbol{z}(t) + \underbrace{\begin{bmatrix} 0 \\ b \end{bmatrix}}_{\boldsymbol{B}} u(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\boldsymbol{D}} d \tag{31}$$

where $b = 1/m$ and $d = -g_x/m$.

The main advantage of this simplified model is that only requires knowledge of the slider mass to design the controllers of the previous sections. Also for the SMC control design, require to add the new perturbation, due the gravity acceleration, to the vector $\boldsymbol{D}(\boldsymbol{z}_d(t))$.

IV. Experimental Results

In this section we make the experiments and comparisons of the optimal tracking control or Linear Quadratic Tracking (LQT) and the SMC using the linearized dynamics and the slider dynamics of an inverted Whitworth mechanism.

A. Inverted Whitworth mechanism dynamics

The inverted Whitworth mechanism scheme is shown in Fig. 1 and its prototype is shown in Fig. 2, where the real time environment is Matlab/Simulink® using a Sensoray model 626 for data acquisition. For the slider position measures it is used a US-digital strip sensor with a resolution of 300 cpi.

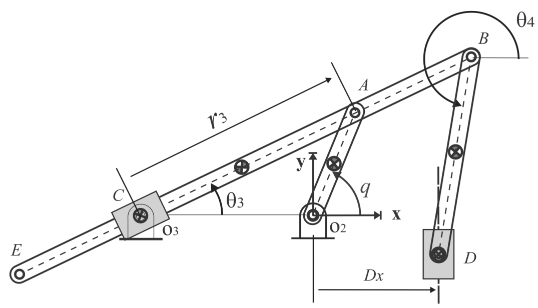

Fig. 1. Inverted Whitworth mechanism

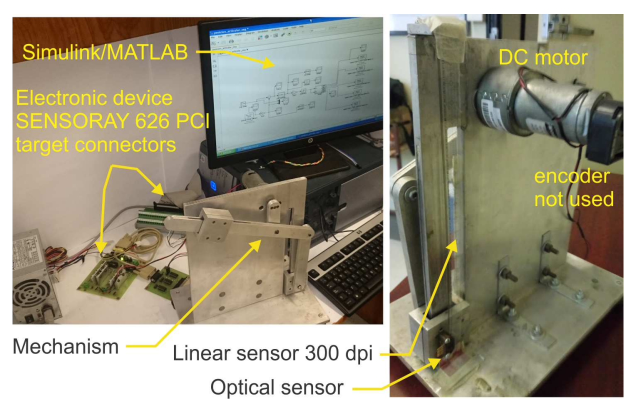

Fig. 2. Inverted Whitworth mechanism prototype

The linearized dynamics of the inverted Whitworth mechanism in task space is:

$$\ddot{y} = -0.0281396y + 0.253427u.$$

Note that it is required the complete knowledge of the mechanism dynamics. In state space is expressed as:

$$\dot{\boldsymbol{z}}(t) = \begin{bmatrix} 0 & 1 \\ -0.0281396 & 0 \end{bmatrix} \boldsymbol{z}(t) + \begin{bmatrix} 0 \\ 0.253427 \end{bmatrix} u(t)$$

For our approach, the generalized coordinate vector is given by $\boldsymbol{q}' = [q, r_3, \theta_3, \theta_4, y]^\top$, the extended dynamic model is

$$\boldsymbol{M}'(\boldsymbol{q}') = \begin{bmatrix} M_{11} & 0 & 0 & 0 & 0 \\ 0 & M_{22} & 0 & M_{24} & 0 \\ 0 & 0 & M_{33} & M_{34} & 0 \\ 0 & M_{24} & M_{34} & M_{44} & 0 \\ 0 & 0 & 0 & 0 & m_5 \end{bmatrix}$$

$$\boldsymbol{C}'(\boldsymbol{q}', \dot{\boldsymbol{q}}') = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -C_{23} & C_{24} & 0 \\ 0 & C_{23} & C_{33} & C_{34} & 0 \\ 0 & C_{42} & C_{43} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{G}(\boldsymbol{q}') = \begin{bmatrix} G_1 & G_2 & G_3 & G_4 & m_5 g \end{bmatrix}^\top$$

$$\boldsymbol{\rho}(\boldsymbol{q}') = \begin{bmatrix} 1 \\ \rho_{r_3} \\ \rho_{\theta_3} \\ \rho_{\theta_4} \\ \frac{AB\sin(2-2\theta_3+\theta_4)-(AB+2r_3)\sin(q-\theta_4)}{2r_2^{-1}r_3\sin(\theta_4)} \end{bmatrix}$$

where m_5 is the slider mass, $g_x = m_5 g$, and $\rho_x(\boldsymbol{q}') = \frac{AB\sin(2-2\theta_3+\theta_4)-(AB+2r_3)\sin(q-\theta_4)}{2r_2^{-1}r_3\sin(\theta_4)}$. The Jacobian ρ_x can be compensated by using the compensator of our previous work [25]. Finally we have that the slider dynamic equation is:

$$m_5\ddot{y} + m_5 g = u \tag{32}$$

that is a simple double integrator system with a gravity perturbation. The value of the slider mass is $m_5 = 4.5\times 10^{-4}$. In state space:

$$\dot{\boldsymbol{z}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{z}(t) + \begin{bmatrix} 0 \\ 2222.22 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ -9.81 \end{bmatrix}$$

B. Controllers design

The desired trajectory is given by the following expression:

$$y_d(t) = -0.19 + 0.03\sin\left(\frac{\pi}{2}t\right) \tag{33}$$

The desired reference in state space is given by $\dot{\boldsymbol{z}}_d = [y_d, \dot{y}_d]^\top$. For the LQT it is proposed the weights $\boldsymbol{Q} = \boldsymbol{I}$ and $R = 0.1$. For the SMC it is used the weight $\boldsymbol{Q} = \mathrm{diag}(1, 0.1)$. It is compared the solutions of the LQT and SMC using the linearized dynamics and our approach. The controllers for the linearized dynamics are named as LQT 1 and SMC 1, on the other hand, the controllers for our approach are LQT 2 and SMC 2. The gains for the feedback term, sliding hyperplane, and sliding controller are given in Table I. The solution of the tracking problem is given in Fig. 3 and Fig. 4.

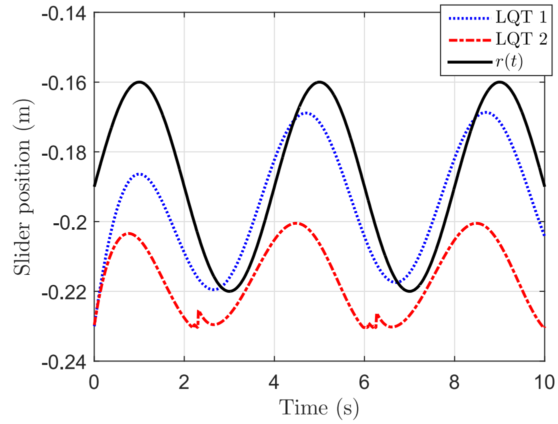

Fig. 3. Position tracking: LQT method

C. Discussion

Fig. 3 shows the performance of the LQT controllers, where it is shown that both controllers do not achieve the control task due the feedforward term. To guarantee the position tracking we require the complete non-linear model of the mechanism, nevertheless it is used a linearized model which affects the accuracy of the output trajectory. The LQT 2 uses a very simple model that affects considerably the output trajectory in comparison to the LQT 1. The feedback term stabilizes the closed-loop system in both cases, but it has been shown that the controller is not robust against modeling error or disturbances.

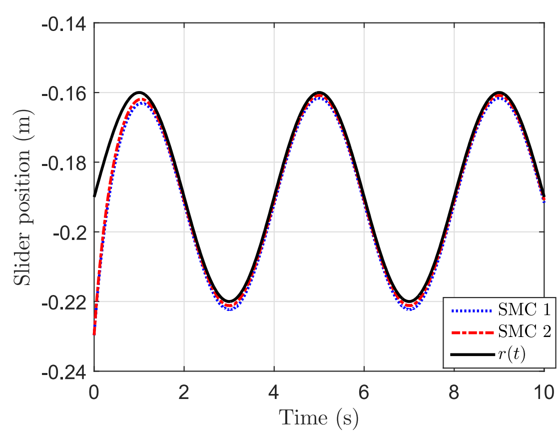

Fig. 4. Position tracking: SMC method

In Fig. 4 is shown the performance of the SMCs. Both controllers achieve the control task in finite time. Here we can see the main advantage of our approach because we can guarantee the tracking task by using a minimal knowledge of the mechanism dynamics, i.e., the slider mass. The feedforward problem of the LQT is avoided and is shown the robustness of the SMC in presence of modeling error and disturbances.

Table I
CONTROLLER GAINS

Dynamics	Controller	$\frac{1}{R}\boldsymbol{H}^\top\boldsymbol{P}$	$\mathbf{C}$	K
$\dot{\mathbf{z}}(t) = \begin{bmatrix} 0 & 1 \\ -0.0281396 & 0 \end{bmatrix}\mathbf{z}(\mathbf{t}) + \begin{bmatrix} 0 \\ 0.253427 \end{bmatrix}\mathbf{u}(\mathbf{t})$	LQT 1	$\begin{bmatrix} 3.1343 & 2.696 \end{bmatrix}$	-	-
	SMC 1	-	$\begin{bmatrix} 12.4781 & 3.9459 \end{bmatrix}$	0.007
$\dot{\boldsymbol{z}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\boldsymbol{z}(t) + \begin{bmatrix} 0 \\ 2222.22 \end{bmatrix}u(t) + \begin{bmatrix} 0 \\ -9.81 \end{bmatrix}$	LQT 2	$\begin{bmatrix} 3.1623 & 4.0404 \end{bmatrix}$	-	-
	SMC 2	-	$\begin{bmatrix} 0.0014 & 0.0004 \end{bmatrix}$	0.0045

V. CONCLUSION

In this paper is presented a simple optimal tracking control for slider-crank mechanisms in task space. By means of the extended dynamic model it is obtained the slider dynamics, which is a linear system with disturbances, without using any method of linearization. It is presented the theory for the LQT and sliding mode controllers design using two model approaches: linearized dynamics and slider dynamics. Experiments are made using an inverted Whitworth mechanism to test our approach. The results show that the accuracy of the closed-loop system using LQT is affected by modeling error, on the other hand, SMC shows good performance and verify that our simple model can guarantee the tracking task with a minimal knowledge of the mechanism dynamics.

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