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Relaxed Conditions for Stability and Stabilization of Takagi-Sugeno Fuzzy Systems with Time-Delay via Non-quadratic Functional

Sara Angulo, Raymundo Márquez, and Miguel Bernal
Department of Electrical and Electronics Engineering
Sonora Institute of Technology, Ciudad Obregón, Sonora, México
email: sara.angulo@potros.itson.edu.mx, raymundo.marquez@itson.edu.mx, miguel.bernal@itson.edu.mx.

***Abstract*—This paper presents more relaxed delay-dependant conditions for stability analysis and stabilization of time-delay nonlinear systems expressed as a Takagi-Sugeno model. These conditions arise using a combination of a non-quadratic Lyapunov-Krasovskii functional with a recent integral inequality that reduces the conservatism of previous approaches. Sufficient conditions for stability and controller design are established in terms of linear matrix inequalities. Some numerical examples are given to show the effectiveness of the proposed approach in comparison with other recent results.**

I. Introduction

It is very common to find time delays in dynamical systems; they are present in engineering, biology, economics, among other areas [1]. The existence of a delay is due to different factors, such as the measurement of system variables, the physical nature of the elements that compose the system, or delays in the transmission signal [2]. Besides, the delay may occur at the input, output or state. The state variables of the dynamic system can represent position, angular speed, temperature, tension, among others. It is also known that delays can produce undesired effects on the performance of systems such as oscillations or instability [3]. For all the reasons above, stability analysis of this class of systems has become very relevant recently. Stability of time-delay systems has been studied from two different points of view: frequency domain approach [4], [3], and the time domain approach. Since the latter is of general application regardless of the linear or nonlinear of the system, more works can be found on it, mainly based on the Lyapunov-Krasovskii functional (LKF) [5], [6], [7] and the Lyapunov-Razumikhin function [8], [9].

Takagi-Sugeno models (TS) have been used in the last decades for analysis and synthesis of nonlinear control systems, including those incorporating time delays. A TS representation can be obtained applying the sector nonlinearity approach in [10], there is no loss of information and the TS model is an *exact representation* within the modeling region. A TS model is formed from a set of linear systems combined with a set of membership functions (MF), which contain the nonlinearities of the model and also satisfy the convex sum property. Taking benefit of the convex structure of the TS model, stability analysis and stabilization has been studied through the Lyapunov direct method and the solution is commonly solved via linear matrix inequalities (LMIs), which can be effectively solved using convex programming methods [11], [12]. In the literature, approaches on stability analysis of TS time-delay systems are classified in two groups, depending on whether the information of the delay is taken into account or not, being the former less conservative than the latter.

The stability analysis usually aims to establish the maximum delay before the system loses stability. There are different methods in literature to study the stability of time-delay systems, some reduce the conservativeness of solutions through integral inequalities as Jensen [13] or Wirtinger [14], [15], [16]; some other results are based on the construction of new Lyapunov-Krasovskii functionals [17], [18]. The stabilization problem has been addressed in a similar way. In [19], the problem is tackled choosing an adequate LKF and including an extension of the Jensen's inequality along with Finsler's lemma. In [18], a LKF which contains the fuzzy line-integral Lyapunov function is employed. A non-quadratic functional approach proposed in [20] has been previously used for stability and stabilization analysis giving less conservative results; see [21], [22]. In this paper, stability analysis and stabilization of nonlinear systems with a constant time delay is considered. The main contribution of this research is to provide more relaxed LMI conditions for stability analysis and control design for time-delay nonlinear systems (TDNS) represented as a TS model; results obtained are based on the combination of a non-quadratic LKF with a recent integral inequality proposed in [23].

This paper is organized as follows: section II provides the methodology to get a TS model of a TDNS and some

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useful properties as well as the notation used throughout this work; the structure of the controller and the form of the non-quadratic functional to be used are also given. Section III presents the main results, one result for stability that arises from a combination of an integral inequality with a non-quadratic Lyapunov-Krasovskii functional; a second one about stabilization, which adds Finsler's lemma to obtain less conservative conditions. Section IV shows with two numerical examples the effectiveness of our proposal. Section V gather the conclusions and future work.

II. Preliminaries

The following affine-in-control time-delay nonlinear system is considered:

$$\dot{x}(t) = f_1(x(t))x(t) + f_2(x(t))x(t-\tau) + f_3(x(t))u(t), \quad (1)$$
$$\phi(\theta) = x(\theta),\ \theta \in [-\tau, 0],$$

where $x(\cdot)$ is the state vector, $f_i(\cdot)$, $i \in \{1,2,3\}$ are sufficiently smooth nonlinear matrix functions, $u(t)$ is the control law, $\tau \in \mathbb{R}^+$ is a constant time delay, $\phi \in \ell([-\tau,0],\mathbb{R}^n)$ is the initial function, $\ell([-\tau,0],\mathbb{R}^n)$ is the Banach space of real continuous functions on the interval $[-\tau, 0]$ with

$$||\phi||_\tau := \sup_{\theta\in[-\tau,0]} ||\phi(\theta)||,$$

where $||\cdot||$ is the Euclidean norm in $\mathbb{R}^n$. A unique solution of the system, $x(t;\phi) \in \mathbb{R}^n$ (state vector), is assumed for each initial condition $\phi \in \ell([-\tau,0],\mathbb{R}^n)$ and $t \geq 0$. A segment of solution $x(t;\phi)$ is represented by $x_t(\phi) := \{x(t+\theta;\phi) : \theta \in [-\tau,0]\} \subset \mathbb{R}^n$. For simplicity, $x(t)$ and x_t are employed instead of $x(t;\phi)$ and $x_t(\phi)$.

An exact TS fuzzy model of a TDNS can be obtained using the sector nonlinearity approach proposed in [10]. This methodology consists in performing the following steps:

1) Group in the premise vector function $z(x(t))$ the p non-constant terms identified in $f_i(x)$, $i \in \{1,2\}$. $z(x(t))$ is considered smooth and bounded in a compact set $\mathcal{C}$ including the origin, i.e.,
$$z(x) = \begin{bmatrix} z_1(x) & z_2(x) & \cdots & z_p(x) \end{bmatrix}^T,$$
with
$$z_j : \mathbb{R}^n \to [z_j^0, z_j^1] \subset \mathbb{R}, j \in \{1,2,\dots,p\},$$
where $z_j^0 = \min_{x(t)\in\mathcal{C}} z_j(x)$, $z_j^1 = \max_{x(t)\in\mathcal{C}} z_j(x)$.
2) Build the following pairs of weighting functions (WFs) for each nonlinear term in $z(x)$:
$$\omega_0^j(x) = \frac{z_j^1 - z_j(x)}{z_j^1 - z_j^0},\ \omega_1^j(x) = 1 - \omega_0^j(x),$$
for $j \in \{1,2,\dots,p\}$.
3) Set the following $r = 2^p$ membership functions (MFs):
$$\begin{aligned} h_i(z(x)) &= h_{1+i_1+i_2\times 2+\cdots+i_p\times 2^{p-1}}(z(x)), \\ &= \prod_{j=1}^{p} \omega_{i_j}^j(z_j), i \in \{1,2,\dots,r\}, i_j \in \{0,1\}. \end{aligned}$$
Note that $\sum_{i=1}^r h_i(\cdot) = 1$, $h_i(\cdot) \geq 0$, i.e., the MFs h_i hold the convex sum property in $\mathcal{C}$. This fact is the key to apply the direct Lyapunov method (or its extension in the Lyapunov-Krasovskii context) in order to obtain a set of LMI conditions for stability analysis and stabilization.
4) Finally, rewrite the TDNS in (1) as an exact TS representation defined by:
$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(x))(A_i x(t) + A_{di}x(t-\tau) + B_i u(t)), \quad (2)$$
$$\phi(\theta) = x(\theta), \theta \in [-\tau, 0],$$
where $A_i = f_1(x(t))|_{h_i=1}$, $A_{di} = f_2(x(t))|_{h_i=1}$, and $B_i = f_3(x(t))|_{h_i=1}$, $i = 1,2,\dots,r$, with $r = 2^p \in \mathbb{N}$, are matrices of proper size.

A. Notation and properties

In this paper, an asterisk $(*)$ denotes the transpose of the terms on its left-hand side for inline expressions and the transpose of its symmetric block-entry for matrix expressions; for a symmetric matrix $\mathcal{M}$, $\mathcal{M} > 0$ ($\mathcal{M} < 0$) means that $\mathcal{M}$ is positive definite (negative definite, respectively).

For simplicity, when expressions involving convex sums appear the following shorthand notation will be adopted whenever considered appropriate: $\Upsilon_h = \sum_{i=1}^r h_i(z(t))\Upsilon_i$ for single sums, $\Upsilon_{h^-} = \sum_{i=1}^r h_i(z(t-\tau))\Upsilon_i$ for single convex sums with time delay τ, $\Upsilon_{h^s} = \sum_{i=1}^r h_i(z(s))\Upsilon_i$ for single convex sums depending on s, $\dot{\Upsilon}_v = \frac{d}{dt}\left(\sum_{i=1}^r v_i(z(t))\Upsilon_i\right)$ for the time-derivative of a convex sum, and $\Upsilon_{hv} = \sum_{i=1}^r \sum_{j=1}^r h_i(z(t))v_j(z(t))\Upsilon_{ij}$ for double convex sums with different sets of MFs. When convenient, arguments will be omitted.

According to the previous notation, the TS model (2) is rewritten as:

$$\dot{x} = A_h x + A_{dh} x_\tau + B_h u, \quad (3)$$
$$\phi(\theta) = x(\theta), \theta \in [-\tau, 0],$$

with $x_\tau = x(t-\tau)$.

The following lemmas and properties will be applied to obtain the main results of this work:

Property 1. (Schur complement lemma) [11]: Let $P \in \mathbb{R}^{m\times m} : P = P^T > 0$, $X \in \mathbb{R}^{m\times n}$ a full rank matrix, and $Q \in \mathbb{R}^{n\times n}$, then:

$$\left\{ \begin{array}{l} Q - X^T P^{-1} X > 0 \\ P > 0 \end{array} \right. \Leftrightarrow \begin{bmatrix} Q & X^T \\ X & P \end{bmatrix} > 0. \quad (4)$$

Property 2. [23]: Let x be a differentiable function: $[\alpha,\beta] \to \mathbb{R}^n$. For symmetric matrices $\mathcal{R} \in \mathbb{R}^{4n\times n}$, the following inequality holds:

$$-\int_\alpha^\beta \dot{x}^T(s)\mathcal{R}\dot{x}(s)ds \leq \vartheta^T \Omega \vartheta, \quad (5)$$

where

$$\Omega = N_1\Pi_1 + N_2\Pi_2 + N_3\Pi_3 + (*) + \tau\Big(N_1\mathcal{R}^{-1}N_1^T + \frac{1}{3}N_2\mathcal{R}^{-1}N_2^T + \frac{1}{5}N_3\mathcal{R}^{-1}N_3^T\Big),$$

$$\Pi_1 = e_1 - e_2,\ \Pi_2 = e_1 + e_2 - 2e_3,$$

$$\Pi_3 = e_1 - e_2 - 6e_3 + 6e_4,$$

$$\vartheta = \Big[x^T(\beta) \quad x^T(\alpha) \quad \frac{1}{\beta-\alpha}\int_\alpha^\beta x^T(s)ds \quad \frac{2}{(\beta-\alpha)^2}\int_\alpha^\beta\int_\alpha^s x^T(u)duds\Big],$$

with $e_i = \begin{bmatrix}0_{n\times(i-1)n} & I_n & 0_{n\times(4-i)n}\end{bmatrix}$, $i = 1,2,3,4$.

Property 3. (Finsler's lemma) [24]: Let $x \in \mathbb{R}^n$, and matrices $Q = Q^T \in \mathbb{R}^{n\times n}$, and $R \in \mathbb{R}^{m\times n}$ such that $rank(R) < n$, the following statements are equivalent:

$$i)\ x^TQx < 0, \quad \forall x \in \{x \in \mathbb{R}^n : x \neq 0, Rx = 0\}. \tag{6}$$

$$ii)\ \exists X \in R^{n\times m} : Q + XR + R^TX^T < 0. \tag{7}$$

B. Controller design of TS models: control law

In this work, the following control law structure is proposed:

$$u(t) = F_{1hh^-v}Y^{-1}x + F_{2hh^-v}Y^{-1}x_\tau, \tag{8}$$

where

$$F_{1hh^-v} = \sum_{j=1}^r\sum_{k=1}^r\sum_{l=1}^r h_j(z(t))h_k(z(t-\tau))v_l(z(t))F_{1jkl},$$

$$F_{2hh^-v} = \sum_{j=1}^r\sum_{k=1}^r\sum_{l=1}^r h_j(z(t))h_k(z(t-\tau))v_l(z(t))F_{2jkl},$$

with F_{1jkl}, F_{2jkl}, $j,k,l \in \{1,2,\ldots,r\}$, Y as the controller gains of adequate size. This controller considers the value of the delay τ; it is a slight variation of the parallel distributed compensation (PDC) scheme.

After the control law (8) is substituted in the TS model (3), the next closed loop system arises:

$$\dot{x}(t) = G^1_{hhh^-v}x + G^2_{hhh^-v}x_\tau, \tag{9}$$

with $G^1_{hhh^-v} = A_h + B_hF_{1hh^-v}Y^{-1}$ and $G^2_{hhh^-v} = A_{dh} + B_hF_{2hh^-v}Y^{-1}$.

In the stabilization problem, our purpose is to design the gains of the controller F_{1jkl}, F_{2jkl} and Y such that the origin of (9) is asymptotically stable.

C. Non-quadratic Lyapunov-Krasovskii functional

In this paper, the following non-quadratic Lyapunov-Krasovskii functional (NQLKF) is considered:

$$V(x) = V_1(x) + V_2(x) + V_3(x), \tag{10}$$

where

$$V_1 = \bar{x}^TP_v\bar{x}, \quad P_i = P_i^T > 0,$$

$$V_2 = \int_{t-\tau}^t x^T(s)Q_{h^s}x(s)ds, \quad Q_i = Q_i^T > 0,$$

$$V_3 = \int_{t-\tau}^t\int_s^t \dot{x}^T(v)R\dot{x}(v)dvds, \quad R = R^T > 0,$$

with $\bar{x} = \Big[x^T(t) \quad \int_{t-\tau}^t x^T(s)ds \quad \int_{t-\tau}^t\int_{t-\tau}^s x^T(u)duds\Big]^T$, $P_v = \sum_{i=1}^r v_i(z(t))P_i$, $Q_{h^s} = \sum_{i=1}^r h_i(z(s))Q_i$, $P_i \in \mathbb{R}^{3n\times 3n}$, $Q_i \in \mathbb{R}^{n\times n}$, and $R \in \mathbb{R}^{n\times n}$. In [23] a similar form has been proposed for the linear case with a quadratic form which is considered an instance of (10).

The structure of P_v in $V_1(x)$ is provided in [20] with

$$v_i(z(t)) = \frac{1}{\alpha}\int_{t-\alpha}^t h_i(z(\tau))d\tau \geq 0, \alpha > 0. \tag{11}$$

As can be easily verified, the MFs $v_i(\cdot)$ hold the convex-sum property, i.e.:

$$\sum_{i=1}^r v_i(z(t)) = \frac{1}{\alpha}\int_{t-\alpha}^t\left(\sum_{i=1}^r h_i(z(\tau))\right)d\tau = 1. \tag{12}$$

Due to the MFs being smooth and bounded, they are integrable along the trajectories. More relaxed LMI conditions can be achieved with the non-quadratic form of the NQLKF in (10).

Remark 1. The time-derivative of MFs v_i is:

$$\dot{v}_i(z) = \frac{1}{\alpha}\big(h_i(z(t)) - h_i(z(t-\alpha))\big), \tag{13}$$

then, the time-derivative of P_v after considering $\alpha = \tau$, can be written as:

$$\dot{P}_v = \frac{1}{\tau}\left(P_h - P_{h^-}\right), \tag{14}$$

which will be useful for developing the main results in the next section.

III. Main Results

In this section, the LMI conditions obtained for stability analysis and stabilization are based on the NQLKF in (10) altogether with the integral inequality given in (5) and the Finsler's lemma.

A. Conditions for stability

Theorem 1. The origin of the TS model (3) with $u = 0$ and time delay $\tau > 0$, is asymptotically stable if there exist symmetric and definite positive matrices $P_i \in \mathbb{R}^{3n\times 3n}$, $Q_i \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{n\times n}$, $i \in \{1,2,\ldots,r\}$, and arbitrary matrices $N_j \in \mathbb{R}^{4n\times n}$, $j \in \{1,2,3\}$, such that the next LMI conditions hold:

$$\Upsilon^k_{ij} < 0, \quad \forall i,j,k \in \{1,2,\ldots,r\}, \tag{15}$$

where

$$\Upsilon^k_{ij} = \begin{bmatrix}\Psi_{11} & \sqrt{\tau}N_1 & \sqrt{\tau}N_2 & \sqrt{\tau}N_3 \\ * & -R & 0 & 0 \\ * & * & -3R & 0 \\ * & * & * & -5R\end{bmatrix},$$

with

$$\Psi_{11} = \Pi_4^TP_k\Pi_{5i} + N_1\Pi_1 + N_2\Pi_2 + N_3\Pi_3 + (*) + e_1^TQ_ie_1 - e_2^TQ_je_2 + \tau\Gamma_i^TR\Gamma_i + \frac{1}{\tau}\Pi_4^T(P_i - P_j)\Pi_4,$$

$$\Gamma_i = \begin{bmatrix}A_i & A_{di} & 0_n & 0_n\end{bmatrix}, \Pi_4 = \begin{bmatrix}e_1^T & \tau e_3^T & \frac{\tau^2}{2}e_4^T\end{bmatrix}^T,$$

$$\Pi_{5i} = \begin{bmatrix}\Gamma_i^T & \Pi_1^T & \tau(e_3^T - e_2^T)\end{bmatrix}^T.$$

Proof. The time derivative of the NQLKF (10) is:

$$\dot{V}(x) = \bar{x}^T P_v \dot{\bar{x}} + \dot{\bar{x}}^T P_v \bar{x} + \bar{x}^T \dot{P}_v \bar{x} + x^T Q_h x - x_\tau^T Q_{h^-} x_\tau + \tau \dot{x}^T R \dot{x} - \int_{t-\tau}^{t} \dot{x}^T(s) R \dot{x}(s) ds. \tag{16}$$

Applying the integral inequality in property 2, using an auxiliary vector $\xi = \begin{bmatrix} x^T & x_\tau^T & \frac{1}{\tau}\int_{t-\tau}^{t} x^T(s)ds & \frac{2}{\tau^2}\int_{t-\tau}^{t}\int_{t-\tau}^{s} x^T(u)duds \end{bmatrix}^T$, and substituting (14), $\dot{V}(x)$ it can be reordered as:

$$\dot{V}(x) \le \xi^T(t)(\phi_1 + \tau\phi_2 + \phi_3)\xi(t), \tag{17}$$

where

$$\begin{aligned} \phi_1 &= \Pi_4^T P_v \Pi_5 + (*) + e_1^T Q_h e_1 - e_2^T Q_{h^-} e_2 + \tau \Gamma_h^T R \Gamma_h \\ &\quad + \frac{1}{\tau}\Pi_4^T (P_h - P_{h^-})\Pi_4, \\ \phi_2 &= N_1 R^{-1} N_1^T + \frac{1}{3} N_2 R^{-1} N_2^T + \frac{1}{5} N_3 R^{-1} N_3^T, \\ \phi_3 &= N_1 \Pi_1 + N_2 \Pi_2 + N_3 \Pi_3 + (*). \end{aligned}$$

The condition $\dot{V}(x) < 0$ is satisfied if:

$$\xi^T(t)(\phi_1 + \tau\phi_2 + \phi_3)\xi(t) < 0, \tag{18}$$

which after schur complement property can be rewritten as:

$$\xi^T \underbrace{\begin{bmatrix} \phi_1 + \phi_3 & \sqrt{\tau} N_1 & \sqrt{\tau} N_2 & \sqrt{\tau} N_3 \\ * & -R & 0 & 0 \\ * & * & -3R & 0 \\ * & * & * & -5R \end{bmatrix}}_{\Phi_{hh^-}^v} \xi < 0. \tag{19}$$

Conditions (19) are satisfied if $\Phi_{hh^-}^v < 0$; those conditions have different membership functions (h, h^-, and v) then it is possible to remove them directly applying the convex sum property which leads to LMI conditions in (15), thus producing the desired results. □

Remark 2. If the matrices in the NQLKF in (10) are assumed as $P_i = P = P^T > 0$ and $Q_i = Q = Q^T > 0$, the next LMI conditions are obtained following a similar path as in Theorem 1:

$$\Upsilon_i < 0, i \in \{1, 2, \ldots, r\}, \tag{20}$$

where

$$\Upsilon = \begin{bmatrix} \Xi_{11} & \sqrt{\tau} N_1 & \sqrt{\tau} N_2 & \sqrt{\tau} N_3 \\ * & -R & 0 & 0 \\ * & * & -3R & 0 \\ * & * & * & -5R \end{bmatrix},$$

with $\Xi_{11} = \Pi_4^T P \Pi_{5i} + N_1\Pi_1 + N_2\Pi_2 + N_3\Pi_3 + (*) + e_1^T Q e_1 - e_2^T Q e_2 + \tau \Gamma_i^T R \Gamma_i$. Observe that conditions in (20) are a particular case of conditions (19) in Theorem 1.

B. Conditions for stabilization

Theorem 2. The origin of the TS model (3) with time delay $\tau > 0$, under control law (8) is asymptotically stable if for given constants ε_1, ε_2, ε_3 and ε_4, there exist symmetric and definite positive matrices $\bar{R} \in \mathbb{R}^{n\times n}$, $\bar{P}_i \in \mathbb{R}^{3n\times 3n}$, $\bar{Q}_i \in \mathbb{R}^{n\times n}$, $i \in \{1, 2, \ldots, r\}$, and arbitrary matrices $Y \in \mathbb{R}^{n\times n}$, $F_{1jkl} \in \mathbb{R}^{m\times n}$, $F_{2jkl} \in \mathbb{R}^{m\times n}$, $j, k, l \in \{1, 2, \ldots, r\}$, $N_q \in \mathbb{R}^{5n\times n}$, $q \in \{1, 2, 3\}$ such that the following LMI conditions hold:

$$\Upsilon_{ii}^{kl} < 0, \forall (i, k, l) \in \{1, 2, \ldots, r\} \tag{21}$$

$$\frac{2}{r-1}\Upsilon_{ii}^{kl} + \Upsilon_{ij}^{kl} + \Upsilon_{ji}^{kl} < 0, \forall (i, j, k, l) \in \{1, 2, \ldots, r\} \tag{22}$$

where

$$\Upsilon_{ij}^{kl} = \begin{bmatrix} \bar{\Psi}_{11} & \sqrt{\tau}\bar{N}_1 & \sqrt{\tau}\bar{N}_2 & \sqrt{\tau}\bar{N}_3 \\ * & -\bar{R} & 0 & 0 \\ * & * & -3\bar{R} & 0 \\ * & * & * & -5\bar{R} \end{bmatrix}$$

with

$$\begin{aligned} \bar{\Psi}_{11} &= \bar{\Pi}_6^T((A_i Y + B_i F_{1jkl})\bar{e}_1 + (A_{di} Y + B_i F_{2jkl})\bar{e}_2 - Y\bar{e}_5) \\ &\quad + \bar{\Pi}_4^T \bar{P}_k \bar{\Pi}_5 + \bar{N}_1\bar{\Pi}_1 + \bar{N}_2\bar{\Pi}_2 + \bar{N}_3\bar{\Pi}_3 + (*) + \bar{e}_1^T \bar{Q}_i \bar{e}_1 \\ &\quad - \bar{e}_2^T \bar{Q}_j \bar{e}_2 + \tau \bar{e}_5^T \bar{R} \bar{e}_5 + \frac{1}{\tau}\bar{\Pi}_4^T(\bar{P}_i - \bar{P}_j)\bar{\Pi}_4, \\ \bar{e}_i &= \begin{bmatrix} 0_{n\times(i-1)n} & I_n & 0_{n\times(5-i)n} \end{bmatrix}, i = 1, 2, 3, 4, 5, \\ \bar{\Pi}_1 &= \bar{e}_1 - \bar{e}_2, \bar{\Pi}_2 = \bar{e}_1 + \bar{e}_2 - 2\bar{e}_3, \bar{\Pi}_3 = \bar{\Pi}_1 - 6\bar{e}_3 + 6\bar{e}_4, \\ \bar{\Pi}_4 &= \begin{bmatrix} \bar{e}_1^T & \tau\bar{e}_3^T & \frac{\tau^2}{2}\bar{e}_4^T \end{bmatrix}^T, \bar{\Pi}_5 = \begin{bmatrix} \bar{e}_5^T & \bar{\Pi}_1^T & \tau(\bar{e}_3^T - \bar{e}_2^T) \end{bmatrix}^T, \\ \bar{\Pi}_6 &= \bar{e}_1 + \varepsilon_1\bar{e}_2 + \varepsilon_2\bar{e}_3 + \varepsilon_3\bar{e}_4 + \varepsilon_4\bar{e}_5. \end{aligned}$$

Proof. Taking the time-derivative of V in (16), considering an auxiliary vector $\zeta = \begin{bmatrix} x^T & x_\tau^T & \frac{1}{\tau}\int_{t-\tau}^{t} x^T(s)ds & \frac{2}{\tau^2}\int_{t-\tau}^{t}\int_{t-\tau}^{s} x^T(u)duds & \dot{x}^T \end{bmatrix}^T$, and substituting (14), $\dot{V}(x)$ can be expressed as:

$$\dot{V}(x) = \zeta^T(t)\psi_1\zeta(t) - \int_{t-\tau}^{t} \dot{x}^T(s) R \dot{x}(s) ds, \tag{23}$$

where $\psi_1 = \bar{\Pi}_4^T P_v \bar{\Pi}_5 + (*) + \bar{e}_1^T Q_h \bar{e}_1 - \bar{e}_2^T Q_{h^-} \bar{e}_2 + \tau \bar{e}_5^T R \bar{e}_5 + \frac{1}{\tau}\bar{\Pi}_4^T(P_h - P_{h^-})\bar{\Pi}_4$.

Applying property 2 to previous expression with $\vartheta = \zeta$ and matrices $N_j \in \mathbb{R}^{5n\times n}$, $j = \{1, 2, 3\}$, $\dot{V}(x) < 0$ yields:

$$\dot{V}(x) \le \zeta^T(t)(\psi_1 + \tau\psi_2 + \psi_3)\zeta(t) < 0, \tag{24}$$

where $\psi_2 = N_1 R^{-1} N_1^T + \frac{1}{3} N_2 R^{-1} N_2^T + \frac{1}{5} N_3 R^{-1} N_3^T$, $\psi_3 = N_1\Pi_1 + N_2\Pi_2 + N_3\Pi_3 + (*)$.

On the other hand, consider that the TS model in (9) can be rewritten through the auxiliary vector ζ in the following form:

$$\left(G_{hhh^-v}^1 \bar{e}_1 + G_{hhh^-v}^2 \bar{e}_2 - \bar{e}_5\right)\zeta = 0. \tag{25}$$

After applying the Finsler's lemma (property 3) to inequality (24) under equality constraint (25), the following equivalent expression is obtained:

$$\mathcal{V}\left(G_{hhh^-v}^1 \bar{e}_1 + G_{hhh^-v}^2 \bar{e}_2 - \bar{e}_5\right) + (*) + \psi < 0, \tag{26}$$

with $\mathcal{V} = \begin{bmatrix} \mathcal{V}_1^T & \mathcal{V}_2^T & \mathcal{V}_3^T & \mathcal{V}_4^T & \mathcal{V}_5^T \end{bmatrix}^T$ and $\psi = \psi_1 + \tau\psi_2 + \psi_3$.

Pre- and post-multiplying $diag\{Y^T, Y^T, Y^T, Y^T, Y^T\}$ and $diag\{Y, Y, Y, Y, Y\}$, respectively, substituting $\mathcal{V}_1 = Y^{-T}$, $\mathcal{V}_2 = \varepsilon_1 Y^{-T}$, $\mathcal{V}_3 = \varepsilon_2 Y^{-T}$, $\mathcal{V}_4 = \varepsilon_3 Y^{-T}$ and $\mathcal{V}_5 = \varepsilon_4 Y^{-T}$, and after some manipulations, the last inequality yields:

$$\begin{bmatrix} \bar{\Xi}_{11} & \sqrt{\tau}\bar{N}_1 & \sqrt{\tau}\bar{N}_2 & \sqrt{\tau}\bar{N}_3 \\ (*) & -\bar{R} & 0 & 0 \\ (*) & (*) & -3\bar{R} & 0 \\ (*) & (*) & (*) & -5\bar{R} \end{bmatrix} < 0, \tag{27}$$

where

$$\begin{aligned} \bar{\Xi}_{11} = &\bar{\Pi}_6^T((A_h Y + B_h F_{1hh^-v})\bar{e}_1 + (A_{dh} Y + B_h F_{2hh^-v})\bar{e}_2 \\ &- Y\bar{e}_5) + \bar{\Pi}_4^T \bar{P}_v \bar{\Pi}_5 + \bar{N}_1\bar{\Pi}_1 + \bar{N}_2\bar{\Pi}_2 + \bar{N}_3\bar{\Pi}_3 + (*) \\ &+ \bar{e}_1^T \bar{Q}_h \bar{e}_1 - \bar{e}_2^T \bar{Q}_{h^-} \bar{e}_2 + \tau \bar{e}_5^T \bar{R} \bar{e}_5 + \frac{1}{\tau}\bar{\Pi}_4^T(\bar{P}_h - \bar{P}_{h^-})\bar{\Pi}_4, \end{aligned}$$

with $\bar{Q}_h = Y^T Q_h Y$, $\bar{Q}_{h^-} = Y^T Q_{h^-} Y$, $\bar{R} = Y^T R Y$, $\bar{P}_v = diag\{Y^T, Y^T, Y^T\} P_v diag\{Y, Y, Y\}$, and $\bar{N}_i = diag\{Y^T, Y^T, Y^T, Y^T, Y^T\} N_i Y$, for $i = \{1, 2, 3\}$. Applying relaxation (21) and (22) to (27), the set of LMI conditions in Theorem 2 are obtained, thus the proof is concluded. □

IV. EXAMPLES

We consider two numerical examples to show the effectiveness of the main results of this paper; they have been programmed using YALMIP/Sedumi [25] within a MATLAB R2013b platform.

Example 1. Consider the following TS fuzzy system:

$$\dot{x}(t) = \sum_{i=1}^{2} h_i(A_i x(t) + A_{di} x(t-\tau)), \tag{28}$$

with $h_1 = \frac{1}{1+\exp(-2x_1x_2)}$, $h_2 = 1 - h_1$, $A_1 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}$, $A_{d1} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$, $A_2 = \begin{bmatrix} -1 & 0.5 \\ 0 & -1 \end{bmatrix}$, $A_{d2} = \begin{bmatrix} -1 & 0 \\ 0.1 & -1 \end{bmatrix}$.

The maximum delay bound for which the system (28) is asymptotically stable with conditions in Theorem 1 is $2.1602s$. In Table I, our results have been compared with other recent approaches in [21], [26], [27]. Note that the LMI conditions in Theorem 1 give better results than those in [26], [27]. Also, it is possible to observe that the results with Remark 2 are lower than results in theorem 1 because they are a particular case, thus the use of non-quadratic terms in the LKF provides more relaxed conditions.

Example 2. Consider the following time-delay TS model:

$$\dot{x}(t) = \sum_{i=1}^{2} h_i(A_i x(t) + A_{di} x(t-\tau)), \tag{29}$$

with $h_1 = \frac{1}{1+\exp(-2x_1x_2)}$, $h_2 = 1 - h_1$, $A_1 = \begin{bmatrix} 0 & 0.6 \\ 0 & 0.1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $A_{d1} = \begin{bmatrix} 0.5 & 0.9 \\ 0 & 2 \end{bmatrix}$, $A_{d2} = \begin{bmatrix} 0.9 & 0 \\ 1 & 1.6 \end{bmatrix}$, and $B_1 = B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Comparisons between some methodologies on the maximum bound of delay τ are given in Table II. The highest value of delay is $\tau = 1.3536s$ with LMI conditions in Theorem 2 which outperformed former results. Observe that the proposed approach found less conservative LMI conditions.

In this example, a controller as in (8) has been obtained using the constants $\varepsilon_1 = 0.54$, $\varepsilon_2 = 0$, $\varepsilon_3 = -0.13$ and $\varepsilon_4 = 2.43$, due to space limitations, only some of the controller gains are given:

$$F_{1111} = \begin{bmatrix} -0.4344 & -0.4286 \end{bmatrix}, \; F_{1222} = \begin{bmatrix} -0.4349 & -0.4197 \end{bmatrix},$$
$$F_{2111} = \begin{bmatrix} -0.0151 & -0.0048 \end{bmatrix}, \; F_{2222} = \begin{bmatrix} -0.0406 & -0.0121 \end{bmatrix},$$
$$Y = \begin{bmatrix} 0.0216 & 0.0063 \\ 0.0077 & 0.0025 \end{bmatrix}.$$

Without any control law ($u = 0$), the system is unstable. Figure 1 shows the behavior of the states under the control law (8) with a delay $\tau = 1.3536s$, for initial function $x(\theta) = \begin{bmatrix} 2 & 1 \end{bmatrix}$, $\theta \in [-1.3536, 0]$. Notice that the states are asymptotically driven to zero as expected. Figure 2 presents the time evolution of the control action. Using the same controller gains for a delay $\tau = 1.3536s$, an asymptotic behavior remains if we consider a delay $\tau < 1.3536s$ but if we use $\tau > 1.3536s$ this fact is not guaranteed, for instance, with $\tau = 10s$ the closed-loop system is unstable.

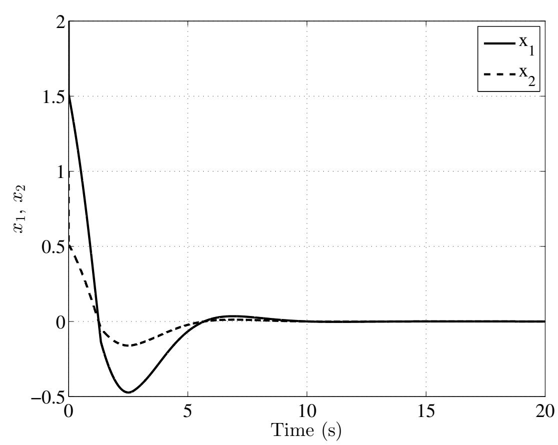

Fig. 1. Behavior of the states in example 2 with delay $\tau = 1.3536$.

TABLE I
MAXIMUM DELAY WITH DIFFERENT APPROACHES: EXAMPLE 1

METHOD	[21]	[26] ($m = 3$)	[27]	Rem. 2 (20)	Th. 2 (15)
τ	1.6010	2.002	2.0689	2.0477	2.1602

TABLE II
MAXIMUM DELAY WITH DIFFERENT APPROACHES: EXAMPLE 2

METHOD	[28] ($m = 3$)	[29] ($m = 3$) (Th. 2)	[22] (Th. 2)	Th. 2 (2)
τ	1.3088	1.3169	1.3332	1.3536

m is the number or partitions in Table I and II [26], [28], [29].

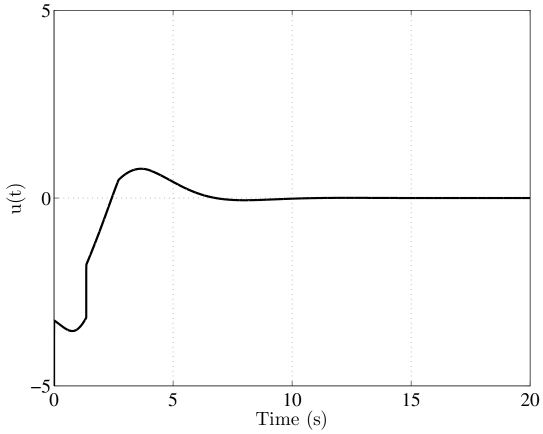

Fig. 2. Control action $u(t)$ for system in example 2 with $\tau = 1.3536$.

V. Conclusion

In this paper a new criteria for stability and stabilization of time-delay nonlinear systems is established via exact Takagi-Sugeno convex structures. It has been proved that the combination of the non-quadratic Lyapunov-Krasovskii functional with a novel integral inequality, reduces the conservatism of recent stability and stabilization results on the subject, thus increasing the feasibility set. All conditions have been formulated in terms of LMIs, which guarantees the numerical treatability of the problem. The effectiveness and applicability of the proposed approaches have been illustrated through some examples found in the literature. Future work is on course to generalize this proposal to the time-varying delay case.

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