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Tracking Regulator Design with Disturbance Rejection to the Reaction–Diffusion Equation

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***Abstract*—In this paper, the geometric regulation theory is used in order to control a distributed parameter system, namely, the reaction–diffusion equation, considering bounded input and output operators, an exogenous system (exosystem) of finite–dimensional to generate the reference and disturbance signals and a feedback control law in function of the state of the exogenous system. In particular, we focus on the State Feedback Regulator Problem (SFRP) which can be solved only if the regulator equations (Francis equations) have solution. For linear distributed parameter systems, these equations can be solved under certain criteria for the eigenvalues from the exogenous system and the transfer function of the system. Under these criteria, we solve the SFRP for the reaction–diffusion equation. The analysis is performed through a series of single–input single–output systems for set–point tracking, set–point disturbance rejection and harmonic tracking, until to finally a tracking regulator with set–point disturbance rejection is designed for the reaction–diffusion equation. Simulation results validate the performance of the regulator.**

***Index Terms*—Distributed parameter systems, disturbance rejection, exosystem, reaction–diffusion equation, state feedback regulator problem, tracking**

I. Introduction

In the output regulation problem the objective is to design a feedback control law that internally stabilizes a control system such that the output of the closed–loop system asymptotically converges to or tracks a reference signal in presence of external disturbances. In the literature, it is assumed that the reference signal and external disturbances are modeled through an exogenous system (exosystem). There are many different approaches in order to solve the output regulation problem. In both finite– and infinite–dimensional cases, the regulator theory has a long history and there is a huge literature about these subjects. The central focus in our work is the so–called State Feedback Regulator Problem (SFRP) when considering that the controller is provided with only full information of the state of the exosystem.

In this work, the methodology here used follows the pioneering work for linear finite–dimensional systems [1], [2], [3], [4]. In [3] was shown that the solvability for the multivariable linear regulator problem corresponds to the solvability of the so–called *regulator* or *Francis equations*, which consist of a system of two linear matrix equations. In [5], necessary and sufficient conditions were given for the solvability of the *Hautus equations* which contain the *Francis equations* as a special case. For finite–dimensional linear systems, the *Hautus conditions* state that no eigenvalue of the exosystem is an invariant zero of the control system [6]. The results of Francis were extended to finite–dimensional nonlinear systems through the center manifold theorem, where necessary and sufficient conditions for the solvability of the regulator problem in terms of the solvability for a pair of nonlinear regulator equations were established [7].

The geometric regulation approach for linear finite–dimensional systems has been extended to linear distributed parameter systems [8] [9] [10] [11] [12]. Control systems whose dynamics are governed by a discrete spectral operator were considered in [10], [11] where the so–called *state operator* satisfies the spectrum decomposition property [13] [14] and a controllability condition, which implies the stabilizability of the control system through a finite–dimensional controller, is determined by the spectrum. Later in [12], the reference signals and disturbances considered in [10], [11] were assumed as generated through a finite–dimensional exogenous system. The case of regulation for linear systems with bounded input and output operators, closely following the development in [12], was considered in [15].

In this work, to be more precise, our application is concerned with the SFRP for a distributed parameter system.

II. Problem Statement

Let us consider a distributed parameter control system in the Hilbert space $\mathcal{Z}$ as given by

$$z_t = Az + B_d d + B_{in} u, \tag{II.1}$$
$$z(0) = z_0, \qquad z_0 \in \mathcal{Z}, \tag{II.2}$$
$$y(t) = Cz(t), \tag{II.3}$$

where z is the state of the system, z_0 denotes the initial condition, z_t is the partial derivative with respect to time, $y(t) \in \mathcal{Y}$ is the measured output and $u(t) \in \mathcal{U}$ is the control signal. $\mathcal{U}$ and $\mathcal{Y}$ are finite– or infinite–dimensional Hilbert input and output spaces, respectively. A is for an unbounded densely defined operator which is assumed as an infinitesimal

generator of a strongly continuous semigroup on $\mathcal{Z}$ [14] [16]. B_{in} refers to a control input operator, $B_d \in \mathcal{L}(\mathcal{U}, \mathcal{Z})$ refers to a disturbance input operator, $d(t)$ denotes a disturbance, and $C \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$. $\mathcal{L}(W_1, W_2)$ denotes a set of all bounded linear operators from a Hilbert space W_1 to a Hilbert space W_2 and $\mathcal{L}(W)$ means that $W = W_1 = W_2$.
In some systems governed by partial differential equations, the state operator A is defined in terms of a uniformly elliptic, formally self–adjoint, linear partial differential operator L in the infinite–dimensional Hilbert space $\mathcal{Z} = L^2(\Omega)$ where Ω is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary. To the second order case,

$$L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial}{\partial x_i}\right) - a(x) \tag{II.4}$$

where a, $a_{ij} \in C^\infty(\overline{\Omega})$, $\overline{\Omega}$ is the closure of Ω, $a_{ij}(x) = a_{ji}(x)$, and $a(x) \geq 0$ for all $x \in \Omega$. Uniform ellipticity means that there exist constants $0 < c_1 < c_2 < \infty$ such that

$$c_1|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq c_2|\xi|^2 \quad \forall\ \xi \in \mathbb{R}^n$$

and $x \in \overline{\Omega}$, where $|\cdot|$ is for the Euclidean norm in $\mathbb{R}^n$. The state operator A is a restriction of the operator L to a dense domain $D(A) \subset \mathcal{Z}$ in terms of the boundary conditions.
The output operator C is defined as a set of bounded operators C_i as weighted integral of the solution $z(x,t)$, *i.e.*

$$y_i(t) = C_i z = \frac{1}{|\Omega_i|}\int_{\Omega_i} z(x,t)dx \tag{II.5}$$

where

$$|\Omega_i| = \int_{\Omega_i} dx > 0.$$

In the same setting,

$$y_i(t) = C_i z = \langle z, \Psi\rangle = \int_{\Omega} z(x,t)\Psi_i(x)dx$$

for which $\Psi_i \in L^2(\Omega)$. For example, in (II.5) we have

$$\Psi_i(x) = \frac{1}{|\Omega_i|}\mathbf{1}_{\Omega_i}(x)$$

where

$$\mathbf{1}_{\Omega_i}(x) = \begin{cases} 1 & \text{if } x \in \Omega_i, \\ 0 & \text{if } x \notin \Omega_i, \end{cases} \tag{II.6}$$

is the so–called *characteristic (or indicator) function*. Accordingly,

$$y = Cz = [C_1(z)\ C_2(z)\ \cdots\ C_{n_c}(z)]^T.$$

A. Exosystem (Exogenous System)

Let us consider a finite–dimensional linear system, referred to as *exosystem* (*exogenous system*), which generates the reference output $y_r(t)$ and models the disturbance $d(t)$, given by

$$\begin{aligned} \frac{dw(t)}{dt} &= Sw(t), && \text{(II.7)}\\ y_r(t) &= Qw(t), && \text{(II.8)}\\ d(t) &= Pw(t), && \text{(II.9)}\\ w(0) &= w_0, && \text{(II.10)} \end{aligned}$$

where $S \in \mathcal{L}(\mathcal{W})$, $Q \in \mathcal{L}(\mathcal{W}, \mathcal{Y})$, $P \in \mathcal{L}(\mathcal{W}, \mathcal{Z})$, and $\mathcal{W}$ is the exosystem state space.
The stability of the origin for the uncontrolled problem, *i.e.* to the case when $u = 0$ and $d = 0$, is a critical component of the theoretical development [12], [17] of the geometric regulation approach. In this work, by exponential stability we mean what comes next.

A system given in the form

$$\begin{aligned} z_t(x,t) &= Az(x,t), && \text{(II.11)}\\ z(x,0) &= z_0(x), && \text{(II.12)} \end{aligned}$$

is said to be exponentially stable if there are positive constants M and α such that for all $z_0 \in \mathcal{Z}$ the solution z satisfies

$$\|z(\cdot,t)\| = \|e^{At}z_0\| \leq Me^{-\alpha t}\|z_0\| \quad \forall\ t \geq 0. \tag{II.13}$$

In this case it is said that the operator A is stable, which means that it generates an exponentially stable C_0 semigroup in $\mathcal{Z}$. Since our main focus is with tracking and disturbance rejection, we assume that the uncontrolled system is exponentially stable, *i.e.*, the state operator A is stable.
In this work, we are also assuming that the exosystem is neutrally stable [7]. To the linear case, this is equivalent to the origin being stable in a Lyapunov sense which implies that $\sigma(S) \subset i\mathbb{R}$ and S has no nontrivial Jordan blocks. $\sigma(T)$ denotes the spectrum of an operator T and $\rho(T)$ denotes the resolvent set of T. This last assumption is the one given in [12], and covers most common examples including set–point and harmonic tracking.

B. Regulation Problem Statement

Let us define the error signal as given by

$$e(t) = y(t) - y_r(t) = Cz(t) - Qw(t). \tag{II.14}$$

The main goal for the regulation task is to force the measured output to track the reference signal while rejecting the disturbance $d(t)$, *i.e.*, that $e(t) \to 0$ as $t \to \infty$. The main problem here considered is stated as follows.

Problem II.1 *State Feedback Regulator Problem (SFRP). Find a feedback control law*

$$u(t) = \Gamma w(t) \tag{II.15}$$

with $\Gamma \in \mathcal{L}(W,U)$ such that for the closed loop system

$$\begin{aligned} \frac{dz(t)}{dt} &= Az(t) + (B_{in}\Gamma + B_dP)w(t),\\ \frac{dw(t)}{dt} &= Sw(t), \end{aligned}$$

which consists of the interconnection of the control system (II.1)–(II.3) and the exosystem (II.7)–(II.10) with the control law (II.15), the norm of the error

$$\|e(t)\| = \|y(t) - y_r(t)\| = \|Cz(t) - Qw(t)\| \qquad \text{(II.16)}$$

satisfies

$$\|e(t)\| \to 0 \text{ as } t \to \infty, \qquad \text{(II.17)}$$

for any initial condition $z_0 \in \mathcal{Z}$ and $w_0 \in \mathcal{W}$.

In general, $e(t)$ is a finite–dimensional vector so, all l_p norms in (II.17) are equivalent. Since exponential stability on the uncontrolled system is assumed, a state feedback law in the statement of the Problem II.1 is not involved. Although the stabilization problem is important in the regulator theory it is a separate question from the problem on which we are focused in this work, namely, the tracking and disturbance rejection problem. Indeed, the feedback control law is a function of the state of the exosystem. One essential aspect of the geometric regulation theory is that solvability of the SFRP can be characterized in terms of the solvability of a pair of operator equations known as the *Regulator Equations (Francis Equations)*. The main result in the present setting is as follows.

Theorem II.1. *The linear SFRP is solvable if and only if there exist mappings $\Pi \in \mathcal{L}(\mathcal{W}, \mathcal{Z})$ with $rank(\Pi) \subset D(A)$ and $\Gamma \in \mathcal{L}(\mathcal{W}, \mathcal{U})$ satisfying the Regulator Equations (Francis Equations)*

$$\Pi S = A\Pi + B_{in}\Gamma + B_d P, \qquad \text{(II.18)}$$

$$C\Pi = Q. \qquad \text{(II.19)}$$

In addition, if the equations (II.18)–(II.19) have solution, a feedback control law that solves the SFRP is given by

$$u(t) = \Gamma w(t). \qquad \text{(II.20)}$$

Proof: See [12].

III. Tracking Regulator Design for the Reaction–Diffusion Equation

Let us consider a controlled one–dimensional reaction–diffusion equation, generally used to model thermal and mass transport–generation/sink phenomena, given by

$$z_t(x,t) = z_{xx}(x,t) + \lambda z(x,t) + B_d d + B_{in} u, \qquad \text{(III.1)}$$

$$z(0,t) = 0, \qquad \textit{(Dirichlet BC)} \qquad \text{(III.2)}$$

$$z_x(1,t) = 0, \qquad \textit{(Neumann BC)} \qquad \text{(III.3)}$$

$$z(x,0) = \phi(x), \qquad \text{(III.4)}$$

$$y(t) = Cz(t). \qquad \text{(III.5)}$$

The system (III.1)–(III.5) is formulated in the form (II.1)–(II.3) in the Hilbert state space $\mathcal{Z} = L^2(0,1)$. z_x is the partial derivative with respect to space while z_{xx} is for the second partial derivative. In this case, the maximal elliptic operator defined by $L = d^2/dx^2$ in (II.4) with domain $D(L) = H^2(0,1)$, the Sobolev space of functions in $\mathcal{Z}$ with square integrable second derivative, or more explicitly, the functions $\varphi \in \mathcal{Z}$ which together with $d\varphi/dx$ are continuous on $(0,1)$ for which $d^2\varphi/dx^2 \in \mathcal{Z}$. The state operator $A = d^2/dx^2$ is defined as a restriction of L in terms of the boundary conditions. Indeed, A is an unbounded densely defined self–adjoint operator in $\mathcal{Z}$, namely,

$$A\varphi = \varphi'',$$

where

$$D(A) = \{\varphi \in H^2(0,1) : \varphi(0) = 0, \varphi'(1) = 0\}. \qquad \text{(III.6)}$$

The spectrum of A

$$\sigma(A) = \{\lambda_k\}_{k=0}^{\infty}, \quad \lambda_k = -\mu_k^2, \quad \mu_k = \left(k - \frac{1}{2}\right)\pi,$$

is purely discrete with a corresponding set of orthonormal eigenvectors

$$\varphi_k(x) = \sqrt{2}\sin(\mu_k x), \quad k = 1, 2, \ldots .$$

The state operator A generates a strongly continuous (analytic) semigroup in terms of the eigenfunction expansion

$$e^{At}\varphi = \sum_{j=0}^{\infty} e^{\lambda_j t}\langle \varphi, \varphi_j\rangle \varphi_j.$$

Moreover, we are considering a single–input single–output system with bounded disturbance B_d, scalar input and output B_{in} and C, respectively, so that $\mathcal{Y} = \mathcal{U} = \mathbb{R}$ [15].

1) *Input operator:* The input to the transport reaction system is spatially uniform over a small interval about a fixed point $x_{in} = x_0 \in (0,1)$.

$$B_{in}u = b(x)u, \qquad \text{(III.7)}$$

$$b(x) = \frac{1}{2\nu_0}\mathbf{1}_{[x_0-\nu_0, x_0+\nu_0]}(x). \qquad \text{(III.8)}$$

The input operators B_d and B_{in} are given as

$$B_d d(t) = \sum_{j=1}^{n_d} B_d^j d_j(t), \quad B_{in}u(t) = \sum_{j=1}^{n_{in}} B_{in}^j u_j(t),$$

where $d_j(t)$ and $u_j(t)$ are scalar disturbances and control inputs, respectively. B_{in}^j and $B_d^j(x)$ are defined as the characteristic function of a bounded subset of Ω, namely

$$B_{in}^j(x) = \frac{1}{|\Omega_j|}\mathbf{1}_{\Omega_j}(x).$$

In order to guarantee that $B_{in}^j \in \mathcal{Z}$, it is assumed that $|\Omega_j| > 0$.

2) *Output operator:* The measured output is the average transport reaction over a small interval about a point $x_{out} = x_1 \in (0,1)$.

$$C\phi = \int_0^1 c(x)\phi(x)dx,$$

$$c(x) = \frac{1}{2\nu_1}\mathbf{1}_{[x_1-\nu_1, x_1+\nu_1]}(x). \qquad \text{(III.9)}$$

Since $C\varphi = \langle \varphi, c\rangle$, C is a bounded linear observation functional on $\mathcal{Z}$.

3) *Disturbance operator:* In this problem a constant disturbance $d(t) = M_d \in \mathbb{R}$ that enters across the entire interval is considered so that $B_d = 1$.

The design objective is to find a feedback control law (II.20) that will force the output $y(t)$ to asymptotically track a reference trajectory $y_r(t) = A_r \sin(\alpha t)$. In this case, we define the exosystem in (II.7) to be governed by the system

$$S = \begin{bmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad w(0) = \begin{bmatrix} 0 \\ A_r \\ M_d \end{bmatrix}. \tag{III.10}$$

The solution of the previous system is given as

$$w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} A_r \sin(\alpha t) \\ A_r \cos(\alpha t) \\ M_d \end{bmatrix},$$

with $Q = [1, 0, 0]$ and $P = [0, 0, 1]$.
Next, let us analyze two special cases about reference signal tracking and disturbance rejection for single–input single–output systems.

A. Set–Point Tracking

For the set point control problem without a disturbance we want to track a constant $y_r = M_r \in \mathbb{R}$. In this case,

$$w_t = Sw = 0, \ w(0) = M_r \ \Rightarrow \ w(t) = M_r.$$

So, $S = 0$, $Q = 1$, and $P = 0$. Consequently,

$$0 = A\Pi + B_{in}\Gamma, \quad C\Pi = 1.$$

From these latter, the first equation implies

$$\Pi = -A^{-1}B_{in}\Gamma.$$

Then, from the second equation, when substituting this last result it becomes

$$1 = C\Pi = C(-A)^{-1}B_{in}\Gamma = G(0)\Gamma,$$

where the notation $G(s) = C(sI - A)^{-1}B_{in}$ for $s \in \rho(A)$ has been used. At last, assuming that $G(0) \neq 0$,

$$\Gamma = G(0)^{-1}. \tag{III.11}$$

B. Set–Point Disturbance Rejection

Let us consider the case for which we want to reject a constant disturbance $d = M_d \in \mathbb{R}$. To do this,

$$w_t = Sw = 0, \ w(0) = M_d \ \Rightarrow \ w(t) = M_d.$$

So, $S = 0$, $Q = 0$, and $P = 1$. Consequently, the regulator equations now become

$$0 = A\Pi + B_d + B_{in}\Gamma, \quad C\Pi = 0.$$

This last system can be easily solved for Γ by

$$\Gamma = -\frac{C(-A)^{-1}B_d}{C(-A)^{-1}B_{in}} = -\frac{G_{B_d}(0)}{G(0)}, \tag{III.12}$$

where the notation $G_{B_d}(s) = C(sI - A)^{-1}B_d$ has been used, assuming the non–resonance condition $G(0) \neq 0$.

C. Harmonic Tracking

For the case in which is desirable to track a harmonic signal $y_r = A_r \sin(\alpha t)$, we define an exosystem given as

$$w_t = Sw, \quad w(0) = \begin{bmatrix} 0 \\ A_r \end{bmatrix}, \quad S = \begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix},$$

whose solution is given by

$$w(t) = \begin{bmatrix} A_r \sin(\alpha t) \\ A_r \cos(\alpha t) \end{bmatrix}.$$

Thus, we take $Q = [1, 0]$ and $P = [0, 0]$ such that

$$y_r(t) = Qw = A_r \sin(\alpha t).$$

In this case, $\mathcal{W} = \mathbb{R}^2$ so then $\Pi = [\Pi_1, \Pi_2]$ and $\Gamma = [\Gamma_1, \Gamma_2] \in \mathbb{R}^2$ with $\Pi_j \in \mathcal{Z}$.
The regulator equations applied to a general vector $w = [w_1 \ w_2]^T \in \mathcal{W}$ give the following system

$$\Pi S w = A\Pi w + B_{in}\Gamma w, \quad C\Pi w = Qw.$$

The first regulator equation can be written as

$$\alpha\Pi_1 w_2 - \alpha\Pi_2 w_1 = A\Pi_1 w_1 + A\Pi_2 w_2 + B_{in}\Gamma_1 w_1 + B_{in}\Gamma_2 w_2.$$

Since this last equation has to hold for all w, consider the special case for which $w_1 = 1$ and $w_2 = 0$ and then for which $w_1 = 0$ and $w_2 = 1$ yielding

$$-\alpha\Pi_2 - A\Pi_1 = B_{in}\Gamma_1,$$
$$\alpha\Pi_1 - A\Pi_2 = B_{in}\Gamma_2.$$

Remarking that the eigenvalues from the exosystem are given by $\lambda = \pm i\alpha$, multiplying the second one of these latter equations by $i = \sqrt{-1}$ to then adding the result to the first one it yields

$$(i\alpha I - A)\Pi_1 + i(i\alpha I - A)\Pi_2 = B_{in} i\Gamma_2 + B_{in}\Gamma_1.$$

Since $i\alpha \notin \rho(A)$, applying $(i\alpha I - A)^{-1}$ to both sides of this last equation it results

$$\Pi_1 + i\Pi_2 = (i\alpha I - A)^{-1}B_{in}(\Gamma_1 + i\Gamma_2).$$

Then, applying C to both sides of the previous equation and recalling that the second regulator equation implies

$$C\Pi_1 = 1, \quad C\Pi_2 = 0, \quad C\Pi w = Qw, \quad Q = [1, 0],$$

consequently,

$$1 = C(i\alpha I - A)^{-1}B_{in}(i\Gamma_2 + \Gamma_1) = G(i\alpha)(\Gamma_1 + i\Gamma_2).$$

Rewriting this last expression in terms of $G(i\alpha) = \mathtt{Re}(G(i\alpha)) + i\mathtt{Im}(G(i\alpha))$ it becomes

$$1 = (\mathtt{Re}(G(i\alpha)) + i\mathtt{Im}(G(i\alpha)))(\Gamma_1 + i\Gamma_2).$$

From the above, equating the real and imaginary parts,

$$1 = (\mathtt{Re}(G(i\alpha))\Gamma_1 - \mathtt{Im}(G(i\alpha))\Gamma_2,$$
$$0 = (\mathtt{Im}(G(i\alpha))\Gamma_1 + \mathtt{Re}(G(i\alpha))\Gamma_2.$$

Hence,

$$\Gamma_1 = \frac{\mathtt{Re}(G(i\alpha))}{|G(i\alpha)|^2}, \qquad \Gamma_2 = -\frac{\mathtt{Im}(G(i\alpha))}{|G(i\alpha)|^2}.$$

Accordingly, the desired control is given by

$$\Gamma = [\Gamma_1, \Gamma_2] = [\mathtt{Re}(G(i\alpha)^{-1}), \mathtt{Im}(G(i\alpha)^{-1})]. \quad \text{(III.13)}$$

This last result follows from the fact that

$$G(i\alpha)^{-1} = \frac{1}{G(i\alpha)} = \frac{\overline{G(i\alpha)}}{|G(i\alpha)|^2} = \frac{\mathtt{Re}(G(i\alpha)) - i\mathtt{Im}(G(i\alpha))}{|G(i\alpha)|^2}.$$

It must be noticed that there is a non–resonance condition for solvability, namely, $G(i\alpha) \neq 0$.

In order to solve the tracking problem for the reaction–diffusion equation, we have defined above all the necessary properties to find the desired control (II.20).

In our case, the regulator equations take the form

$$\begin{aligned} \Pi S w &= A_\lambda \Pi w + B_d P w + B_{in} \Gamma w, \\ C\Pi w &= Q w = w_1, \end{aligned}$$

with $A_\lambda = (A + \lambda)$, where

$$\Pi = [\Pi_1, \Pi_2, \Pi_3], \ w(t) = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} A_r \sin(\alpha t) \\ A_r \cos(\alpha t) \\ M_d \end{bmatrix},$$

and

$$S = \begin{bmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ Q = [1, 0, 0], \ P = [0, 0, 1].$$

The block diagonal structure of the exosystem allow us to decouple the regulator equations into two parts. The first one, corresponding to the harmonic tracking, as given by

$$\Pi^\alpha S^\alpha w^\alpha = A_\lambda \Pi^\alpha w^\alpha + B_{in} \Gamma^\alpha w^\alpha, \quad C\Pi^\alpha w^\alpha = w_1$$

where

$$\Pi^\alpha = [\Pi_1, \Pi_2], \ S^\alpha = \begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix}, \ w^\alpha = [w_1, w_2]^T,$$

$$Q^\alpha = [1, 0], \ P^\alpha = [0, 0].$$

The second one, corresponding to the rejection of a constant disturbance, as given by

$$0 = A_\lambda \Pi_3 w_3 + B_d w_3 + B_{in} \Gamma_3 w_3, \ C\Pi_3 w_3 = Q_3 w_3 = 0,$$

with $Q_3 = 0$ and $P_3 = 1$.

Hence, the regulator equations for the harmonic tracking become exactly as those in Section III-C, whose solution for Γ^α is given in (III.13), while the regulator equations for the set point disturbance rejection become exactly as those in Section III-B whose solution for Γ_3 is given by (III.12). Combining these two solutions, we have

$$\begin{aligned} \Gamma &= [\Gamma_1, \Gamma_2, \Gamma_3] \\ &= [\mathtt{Re}(G(i\alpha)^{-1}), \mathtt{Im}(G(i\alpha)^{-1}), -G(0)^{-1} G_{B_d}(0)]. \end{aligned}$$

Thus, the analytic computation of the control is reduced to finding an explicit formula for the transfer function $G(s)$.

IV. Simulation results

In our numerical simulation, we have set $M_d = 1$, $A_r = 1$, $\alpha = 2$, $x_0 = 0.75$, $x_1 = 0.25$ and $\nu_0 = \nu_1 = 0.25$. Figures 1–3 show the controlled output $y(t) = Cz$ with initial condition $\varphi(x) = 4\cos(\pi x)$ where the reference signal has been selected as $y_r(t) = A_r \sin(\alpha t)$. Also, the error signal $e(t)$ has been included. Simulation results show that the error signal approaches zero as time tends to infinity. Moreover, it can be seen that for small λ values the convergence of the controlled output to the reference signal is faster. The corresponding solution surface for every λ value is shown through figures 4–6. In mass balance, where the diffusion is produced by transport and the term λz is present then if λ is greater than zero this term can be interpreted as generation due to reaction whereas for the case where λ is less than zero it can be explained as inhibition by conversion od species by the reaction.

V. Conclusion

In this work, the geometric regulation theory is used in order to solve the tracking regulation problem with disturbance rejection for the reaction–diffusion equation. The reaction–diffusion equation is formulated in the regulator equations form to then be solved under criteria based on the eigenvalues of the exosystem and the transfer function of the system. Simulation results suggest the desired result for the error signal, *i.e.*, the error approaches zero as time tends to infinity. From the above, we conclude that our proposal performs well.

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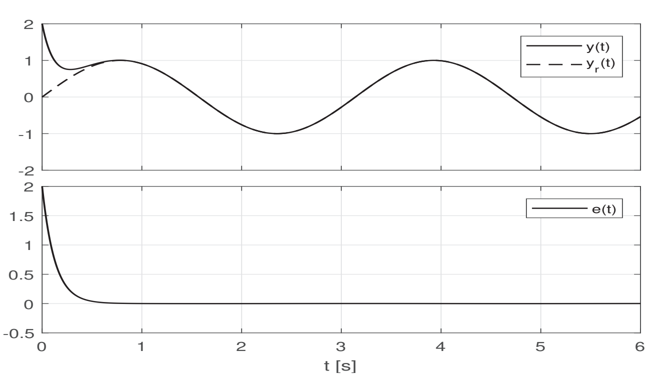

Fig. 1. Dynamics of the regulator for $\lambda = 2$, for whose case $\Gamma_1 = 3.9348$, $\Gamma_2 = 1$, and $\Gamma_3 = -0.5$.

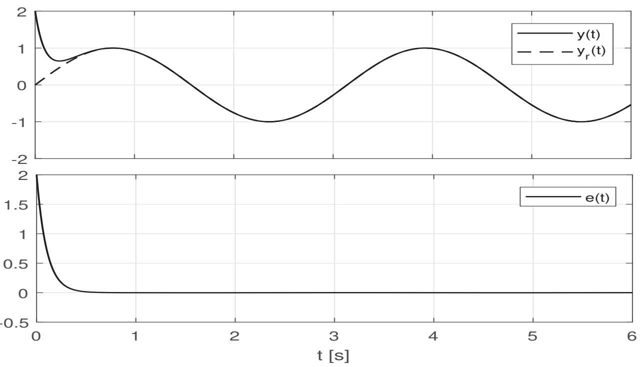

Fig. 2. Dynamics of the regulator for $\lambda = 0$, for whose case $\Gamma_1 = 4.9348$, $\Gamma_2 = 1$, and $\Gamma_3 = -0.5$.

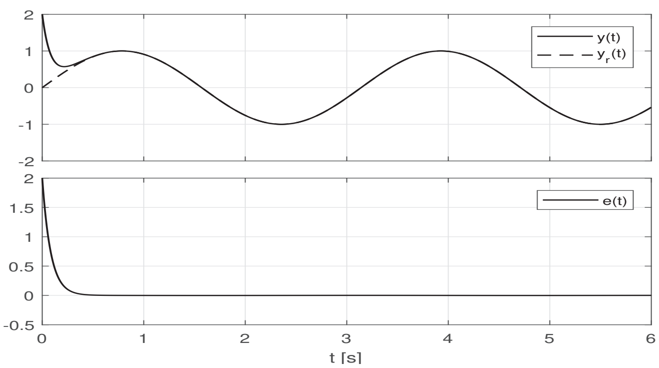

Fig. 3. Performance of the regulator for $\lambda = -2$, for whose case $\Gamma_1 = 5.9348$, $\Gamma_2 = 1$, and $\Gamma_3 = -0.5$.

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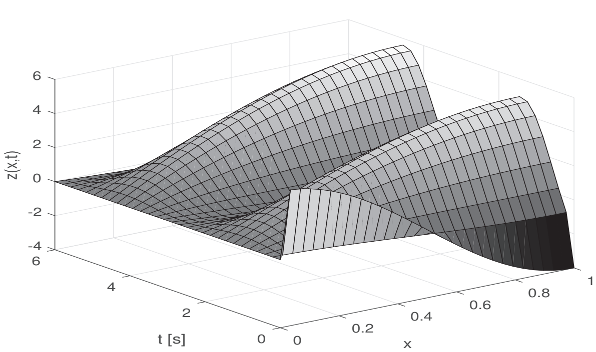

Fig. 4. Plot of the solution surface for $\lambda = 2$.

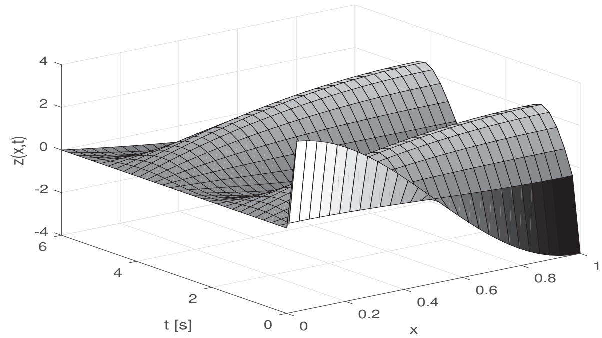

Fig. 5. Solution surface for the case in which the term λz is absent in the reaction–diffusion equation, *i.e.*, for which $\lambda = 0$.

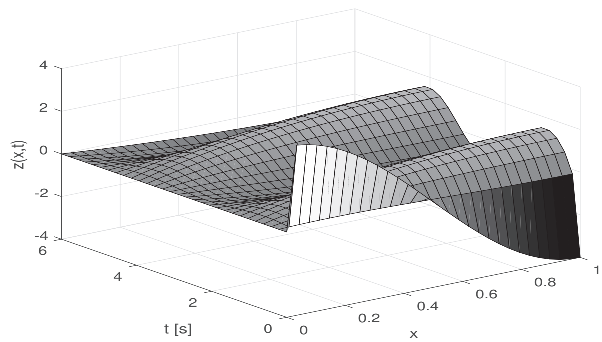

Fig. 6. Plot of the solution surface for $\lambda = -2$.

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